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The leading two loop divergences of the non-leptonic weak chiral Lagrangian

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Abstract. We calculate the leading divergences at NNLO for the octet part of the non-leptonic weak sector of chiral perturbation theory, using renormalization group methods. The role of counterterms which vanish at the equation of motion and their use to simplify the calculation is shown explicitly. The obtained counterterm Lagrangian can be employed to calculate the chiral double log contributions of quantities in this sector, most notably the $K \to \pi\pi$ amplitude. The double log contribution of the latter is discussed in a separate paper.

1 Introduction

We determine the leading divergences at NNLO for the non-leptonic weak chiral Lagrangian which transforms as an octet under the chiral group, extending the NLO calculation of the latter [1,2]. The obtained counterterm Lagrangian can be used to calculate the leading logarithmic contributions, double logs in short, of observables in this sector. These contributions are in particular interesting since all the low energy constants of higher order are unknown in this sector. The calculation of logarithmic contributions provides thus the only way to get a first analytical estimate of the size of the higher order corrections one has to expect. Analogue calculations in the strong sector of chiral perturbation theory (CHPT) have already been worked out [3,4]. The results of this paper are used to calculate the double logs to the $K \to \pi\pi$ amplitude, presented in a separate paper [5]. For the latter amplitude, methods have also been worked out to extract the needed next to leading order (NLO) low energy constants (LEC's) by lattice simulations; however, the proposed approach is rather ambitious [6,7]. Interesting further applications of the results obtained here are for instance the calculation of the double logs of the $K \to \pi\pi\pi$ or the $K \to \pi\gamma\gamma$ amplitude.

The chiral logs are introduced during the process of renormalization [8]. These logarithms, which correspond to the infrared singularities when the masses of the theory approach zero, can produce sizable contributions to observables. Using dimensional regularization, it is straightforward to understand how the leading logs are related to the leading counterterms: while renormalizing the theory, one has to introduce an energy scale μ to ensure the correct dimensions of observables calculated in d-dimensional space-time. In particular, for the divergences generated by loop calculations, this means that they can only show up in the following structures:

$$\mathcal{Q} := \frac{\mu^{d-4}}{(4\pi)^{d/2}}\left(\frac{1}{4-d} - \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right). \qquad (1.1)$$

Table 1. The leading order $\pi\pi$ scattering lengths in two flavor CHPT with the chiral corrections up to NNLO, for some standard values of the NLO LEC's and at a renormalization scale $\mu = 1\,\text{GeV}$. The double logs are included in the NNLO correction

	LO	NLO	NNLO	Double logs
a_0^0	0.156	0.044	0.017	0.013
$a_0^0 - a_0^2$	0.201	0.042	0.016	0.012

To illustrate the order of magnitudes of these chiral corrections, Table 1 displays the various contributions up to NNLO for the $\pi\pi$ scattering lengths in two flavor CHPT, showing that the double log contribution in this case amounts to almost the full NNLO corrections, corresponding to close to 10% of the total result [9].

For the case of three flavor CHPT, we show the chiral corrections up to NNLO to the pion and kaon decay constants and the vector form factor of K_{l3} [10,11,3] in Table 2. One notices that the relative size of the double logs is less pronounced than for two flavor CHPT. Typically, the double logs amount to 20–35% of the total NNLO contributions, corresponding to about 10% of the total corrections to the leading order result. Although the numerical values of these corrections are not too large, one should keep in mind that for applications like chiral extrapolations, the relative size of the double log contributions to the NLO corrections is of importance, which is in

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Table 2. The chiral corrections up to NNLO for the pion and kaon decay constants and the vector form factor of K_{l3}. The values can however vary considerably, depending on the LEC's one employs. The numbers above are calculated with some standard values of the NLO LEC's and all the renormalized NNLO LEC's set to zero at $\mu = 770\,\mathrm{MeV}$

	LO	NLO	NNLO	Double logs
F_π/F_0	1	0.068	−0.172	−0.050
F_K/F_π	1	0.216	0.035	0.06
$f_+(0)[K_{l3}]$	1	−0.023	0.015	0.004

the range of 10% and therefore sizable. Let us in this context again emphasize that while it is certainly appropriate to perform the full two loop calculations in the strong sector, due to the lack of knowledge of the weak LEC's the calculation of the double log contributions is presumably the best one can ever achieve to get an estimate of the size of the NNLO corrections in the weak sector. The outline of this paper is as follows. In Sect. 2 we introduce the notation used and give the needed CHPT Lagrangians. Section 3 provides a very brief overview of the general framework in which this calculation was performed; the generating functional is introduced, and it is shown how one can use the background field method to calculate the counterterms needed to renormalize the latter.

In Sect. 4 we discuss the role of operators which vanish at the solution of the equation of motion (EOM terms) to simplify the calculation of counterterm Lagrangians. One can choose the coefficients of these EOM terms in a way that the sum of all one particle reducible (1PR) topologies at a given $\hbar$-order will not generate any divergences. We pin down these coefficients and henceforth only need to take into account one particle irreducible (1PI) topologies.

In Sect. 5, we sketch very briefly the renormalization group techniques which are employed in the present calculation. The basic result of this section is that one can obtain the $\hbar$-order 2 highest pole counterterm (NNLO) by performing a one loop calculation which uses the $\hbar$-order 1 (NLO) counterterm as input. In the last part, Sect. 6, we illustrate how the concrete calculation works with two simple examples.

2 CHPT Lagrangian

The lowest order chiral Lagrangian which allows for $\Delta S = 1$ strangeness changing interactions is given by (throughout this section we will work in euclidean space-time):

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}_{\mathrm{s}} + \mathcal{L}^{(0)}_{\Delta S=1}, \tag{2.1}$$

which encodes the dynamics of the pseudo-Goldstone bosons in the presence of external source fields s, p, v_μ, a_μ. The first term corresponds to the strong interaction Lagrangian:

$$\mathcal{L}^{(0)}_{\mathrm{s}} = \frac{F_0^2}{4}\big(\langle u_\mu u_\mu\rangle - \langle \chi_+\rangle\big), \tag{2.2}$$

where $\langle\cdot\rangle$ stands for the flavor trace. Further we used:

$$\begin{aligned} u_\mu &= \mathrm{i}\big(u^\dagger(\partial_\mu - \mathrm{i}r_\mu)u - u(\partial_\mu - \mathrm{i}l_\mu)u^\dagger\big),\\ \chi_+ &= u^\dagger\chi u^\dagger + u\chi^\dagger u. \end{aligned} \tag{2.3}$$

A list of additional building blocks used for the Lagrangians of higher order can be found in Appendix D.

The u matrix encodes the octet of the light pseudoscalar bosons in the exponential parametrization:

$$u = \exp\left(\frac{\mathrm{i}\phi}{\sqrt{2}F}\right);$$

$$\phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \tag{2.4}$$

The definitions in (2.3) embody also r_μ and l_μ, the external vector source fields, whereas the scalar counterparts are encoded in χ ($\mathscr{M}$ being the quark mass matrix):

$$\chi = 2B_0\big(\mathscr{M} + s(x) + \mathrm{i}p(x)\big).$$

B_0 is related to the vacuum expectation value of the scalar quark density:

$$\langle 0|\bar{q}q|0\rangle = -F_0^2 B_0(1 + \mathcal{O}(\mathscr{M})).$$

$F_0 \simeq 92.4\,\mathrm{MeV}$ corresponds to the chiral limit value of the pion decay constant.

The Lagrangian which triggers flavor changing processes is given by:

$$\begin{aligned} \mathcal{L}^{(0)}_{\Delta S=1} = CF_0^4\Big(&g_8\langle\Delta_{32}u_\mu u_\mu\rangle - g_8'\langle\Delta_{32}\chi_+\rangle \\ &+ g_{27}t^{ij;kl}\langle\Delta_{ij}u_\mu\rangle\langle\Delta_{kl}u_\mu\rangle\Big) + \text{h.c.}, \end{aligned} \tag{2.5}$$

with $t^{11;23} = t^{13;21} = t^{21;13} = t^{23;11} = 1/3$, $t^{22;23} = t^{23;22} = t^{23;33} = t^{33;23} = -1/6$, and all other t's vanishing.

The constant C,

$$C = -\frac{G_\mathrm{F}}{\sqrt{2}}V_{ud}V^*_{us}, \tag{2.6}$$

renders the coupling constants g_8, g_8' and g_{27} dimensionless.

Δ_{ij} is defined as

$$\Delta_{ij} := u\lambda_{ij}u^\dagger; \quad (\lambda_{ij})_{ab} = \delta_{ai}\delta_{bj}.$$

The first two operators in (2.5), proportional to g_8 and g_8' transform as an octet, $(8,1)$, under the chiral group $\mathrm{SU}(3)_\mathrm{L} \otimes \mathrm{SU}(3)_\mathrm{R}$, whereas the last, proportional to g_{27}, transforms like a 27-plet, $(27,1)$. We neglected a third contribution transforming like $(8,8)$, which takes into account virtual photons.

Since the octet part of (2.5) is believed to be the main source of the $\Delta I = 1/2$ rule, we will only use the latter in our calculation. Furthermore we discard the part proportional to g_8', since it does not contribute to on-shell processes [12–14,1]. Of the remaining operator we only use its CP even part, which we will henceforth denote by $\mathcal{L}_{\mathrm{w}}^{(0)}$:

$$\mathcal{L}_{\mathrm{w}}^{(0)} := CF_0^4 g_8 \langle \Delta u_\mu u^\mu \rangle; \quad \Delta := u\lambda_6 u^\dagger. \tag{2.7}$$

In addition to the lowest order Lagrangians $\mathcal{L}_{\mathrm{s}}^{(0)}$ and $\mathcal{L}_{\mathrm{w}}^{(0)}$ discussed above, we will also use the NLO Lagrangian $\mathcal{L}_{\mathrm{w}}^{(1)}$, introduced in (3.6).

3 The $\hbar$ expansion of the generating functional

In this section we set up some notation and discuss the NLO and NNLO expressions for the generating functional of the non-leptonic weak chiral Lagrangian.

3.1 Notation

The generating functional is defined as the vacuum to vacuum transition amplitude in the presence of sources, collectively denoted by j (throughout this section we will be working in euclidean space-time):

$$\mathrm{e}^{Z[j]/\hbar} = N \int \Pi[\mathcal{D}\phi_i] \mathrm{e}^{-S[\phi,j]/\hbar}. \tag{3.1}$$

$Z[j]$ as well as $S[\phi, j]$ can be split into a strong and weak part:

$$\begin{aligned} Z[j] &= Z_{\mathrm{s}}[j] + Z_{\mathrm{w}}[j]; \\ S[\phi, j] &= S^{\mathrm{s}}[\phi, j] + S^{\mathrm{w}}[\phi, j], \end{aligned} \tag{3.2}$$

analogously to (2.1). These can be expanded in their $\hbar$-order:

$$Z_{\mathrm{s}}[j] = \sum_{n=0}^{\infty} Z_{\mathrm{s}}^{(n)}[j]; \quad S^{\mathrm{s}}[\phi, j] = \sum_{n}^{\infty} S_n^{\mathrm{s}}[\phi, j], \tag{3.3}$$

and similarly for the weak part.

The tree level generating functional $Z^{(0)}[j]$ corresponds to the action with the lowest order chiral Lagrangian, evaluated at the EOM:

$$Z^{(0)}[j] = S^{(0)}[\bar{\phi}, j]. \tag{3.4}$$

One can calculate the separate $\hbar$-order contributions to the generating functional by use of the background field method, which is briefly outlined in Appendix A. The strong and weak non-leptonic chiral actions are expanded in quantum fluctuation fields ξ around a field $\bar{\phi}$, which in the following is assumed to be a solution of the classical equation of motion, i.e. $\bar{\mathfrak{s}}_0^i = 0$. To keep the notation simple, we will henceforth suppress the arguments of $Z[j]$, $S[\phi, j]$ and $\mathcal{L}[\phi]$:

$$\begin{aligned} S^{\mathrm{s}} &= \bar{S}^{\mathrm{s}} + \bar{\mathfrak{s}}^i \xi_i + \frac{1}{2!} \bar{\mathfrak{s}}^{ij} \xi_i \xi_j + \frac{1}{3!} \bar{\mathfrak{s}}^{ij} \xi_i \xi_j \xi_k \\ &\quad + \frac{1}{4!} \bar{\mathfrak{s}}^{ijkl} \xi_i \xi_j \xi_k \xi_l + \mathcal{O}(\xi^5), \\ S^{\mathrm{w}} &= \bar{S}^{\mathrm{w}} + \bar{\mathfrak{w}}^i \xi_i + \frac{1}{2!} \bar{\mathfrak{w}}^{ij} \xi_i \xi_j + \frac{1}{3!} \bar{\mathfrak{w}}^{ijk} \xi_i \xi_j \xi_k \\ &\quad + \frac{1}{4!} \bar{\mathfrak{w}}^{ijkl} \xi_i \xi_j \xi_k \xi_l + \mathcal{O}(\xi^5). \end{aligned}$$

This expansion provides the vertices which will be needed for the calculation of $Z_{\mathrm{w}}^{(2)}$; see (3.9). The Latin indices $i, j, ...$ correspond to an $SU(N)$ index as well as a space-time degree of freedom. The field $\bar{\phi}$ is treated as a background field, and the ξ-field is employed as new integration variable in the path integral; see (3.1). The perturbative evaluation of this "new" generating functional results in vacuum diagrams with respect to the ξ fields.

3.2 NLO: $Z_{\mathrm{w}}^{(1)}$

The counterterms needed to renormalize $Z_{\mathrm{w}}^{(1)}$,

$$Z_{\mathrm{w}}^{(1)} = \frac{1}{2} \bar{\mathfrak{w}}_0^{ij} G_{ij} + \bar{S}_1^{\mathrm{w}}, \tag{3.5}$$

were first calculated by Kambor, Missimer and Wyler [1]. Throughout this paper, we will however use the basis of operators given by Ecker, Kambor and Wyler (EKW) [2], who used the EOM to reduce the former. This Lagrangian assumes the form:

$$\mathcal{L}_{\mathrm{w}}^{(1)} = C g_8 F_0^2 \sum_{i=1}^{37} c_i^{(1)} W_i^{(1)}, \tag{3.6}$$

where the bare LEC's $c_i^{(1)}$ are split into a renormalized and counterterm part:

$$c_i^{(1)} = (\mu c)^{-\varepsilon} \big(c_i^{(1)\mathrm{r}}(\mu, \varepsilon) + a_{1\,i}^{(1)} \Lambda \big). \tag{3.7}$$

μ is the renormalization scale and the constant c parametrizes the regularization prescription ($\ln(c) = \exp(-(\ln(4\pi) + \Gamma'(1) + 1)/2$ for $\overline{\mathrm{MS}}$). In addition we use the notation:

$$\varepsilon := 4 - d \;\; ; \quad \hat{N} := (4\pi)^{-2} \;\; ; \quad \Lambda := \frac{\hat{N}}{\varepsilon}. \tag{3.8}$$

In Table 3, we list the operators of the EKW basis needed for the $K \to \pi\pi$ amplitude at NLO, as well as those which can be shifted by terms which vanish at the equation of motion (marked by an asterisk). The operators given above differ slightly from the original EKW basis, which used $W_{36} = \langle \Delta([\chi_+, \chi_-] + {\chi_+}^2 - {\chi_-}^2) \rangle$. We use the above definition of W_{36} since it simplifies the discussion of the operators which vanish at the solution of the equation of motion in Sect. 4 somewhat.

3.3 NNLO: $Z_{\mathrm{w}}^{(2)}$

At $\hbar$-order 2, we have the following diagrams:

$$Z_{\mathrm{w}}^{(2)} = -\frac{1}{6} \bar{\mathfrak{w}}_0^{ijk} G_{ir} G_{js} G_{kt} \bar{\mathfrak{s}}_0^{rst} + \frac{1}{8} G_{ij} \bar{\mathfrak{w}}_0^{ijkl} G_{kl}$$

Table 3. List of operators needed for the $K \to \pi\pi$ amplitude as well as those which can be shifted by EOM terms, marked by an asterisk. The $a^{(1)}_{1\,i}$ are given for Minkowski space-time. N corresponds to the number of flavors

i	$W_i^{(1)}$	$a^{(1)}_{1\,i}$	$a^{(1)}_{1\,i}(N=3)$	EOM
5	$\langle \Delta\{\chi_+, u^2\}\rangle$	$-N/2$	$-3/2$	
7	$\langle \Delta\chi_+\rangle\langle u^2\rangle$	$3/4+N/8$	$9/8$	
8	$\langle \Delta u^2\rangle\langle \chi_+\rangle$	$-1/4+N/4$	$1/2$	
9	$\langle \Delta[\chi_-, u^2]\rangle$	$-N/4$	$-3/4$	*
10	$\langle \Delta{\chi_+}^2\rangle$	$-3/N+N/4$	$-1/4$	
11	$\langle \Delta\chi_+\rangle\langle\chi_+\rangle$	$-1/2-2/N^2$	$-13/18$	
12	$\langle \Delta{\chi_-}^2\rangle$	0	0	*
13	$\langle \Delta\chi_-\rangle\langle\chi_-\rangle$	0	0	*
23	$\langle \Delta_\mu\{\chi_-, u^\mu\}\rangle$	0	0	*
36	$\langle \Delta[\chi_+, \chi_-]\rangle$	$-1/N+N/4$	$5/12$	*

$$
\begin{aligned}
&+\frac{1}{2}\bar{\mathfrak{w}}_1^{ij}G_{ij}-\frac{1}{2}\bar{\mathfrak{w}}_0^{ik}G_{ij}G_{kl}\bar{\mathfrak{s}}_1^{jl}-\frac{1}{4}G_{ij}\bar{\mathfrak{w}}_0^{ijk}G_{kr}\bar{\mathfrak{s}}_0^{rst}G_{st}\\
&+\frac{1}{4}\bar{\mathfrak{w}}_0^{ij}G_{ik}G_{jl}\bar{\mathfrak{s}}_0^{jkm}G_{mn}\bar{\mathfrak{s}}_0^{mrs}G_{rs}-\frac{1}{2}\bar{\mathfrak{w}}_1^{i}G_{ir}\bar{\mathfrak{s}}_0^{rst}G_{st}\\
&-\frac{1}{2}G_{ij}\bar{\mathfrak{w}}_0^{ijk}G_{kr}\bar{\mathfrak{s}}_1^{r}+\frac{1}{2}\bar{\mathfrak{w}}_0^{ik}G_{ij}G_{kl}\bar{\mathfrak{s}}_0^{jlm}G_{mn}\bar{\mathfrak{s}}_1^{n}-\bar{\mathfrak{w}}_4^{i}G_{ir}\bar{\mathfrak{s}}_1^{r}\\
&+\bar{S}_2^{\mathrm{w}}+\mathcal{O}(G_{\mathrm{F}}^2),
\end{aligned}
\tag{3.9}
$$

where G_{ij} is the propagator corresponding to the ξ field, whose ultraviolet divergent part is provided in Appendix B. The subscript of the vertices denotes their $\hbar$-order. The diagrams corresponding to (3.9) are drawn in Fig. 1.

The NNLO Lagrangian $\mathcal{L}_{\mathrm{w}}^{(2)}$, represented by diagram k in Fig. 1, has to cancel the divergences which are generated from the loop part of $Z_{\mathrm{w}}^{(2)}$. It takes the form:

$$
\mathcal{L}_{\mathrm{w}}^{(2)}=Cg_8\sum_i c_i^{(2)}W_i^{(2)},
\tag{3.10}
$$

with the bare LEC's:

$$
c_i^{(2)}=(\mu c)^{-2\varepsilon}\Big(c_i^{(2)\mathrm{r}}(\mu,\varepsilon)+a^{(2)}_{1\,i}\big(\boldsymbol{c}^{(1)}(\mu,\varepsilon)\big)\Lambda+a^{(2)}_{2\,i}\Lambda^2\Big).
\tag{3.11}
$$

3.3.1 The connection between $\mathcal{A}_2^{(2)}$ and the double chiral logs

The highest pole of $\mathcal{L}^{(2)}$, $\mathcal{A}_2^{(2)}:=\sum a^{(2)}_{2\,i}W_i^{(2)}$, can be used to calculate the double chiral logs which are generated from genuine two loop diagrams of a process under consideration. Let us outline how this works. The sum of all diagrams in Fig. 1 with the exception of the counterterm diagram k will result in an expression which is proportional to the square of $\mathcal{Q}$, see (1.1), plus other contributions which are not related to double logs (abbreviated by the dots):

$$
\begin{aligned}
&Cg_8\sum_i\alpha_i W_i^{(2)}\mathcal{Q}^2+\ldots\\
&=Cg_8\sum_i\alpha_i W_i^{(2)}\mu^{-2\varepsilon}\\
&\quad\times\left(\Lambda^2-\Lambda\hat{N}\log\left(\frac{m^2}{\mu^2}\right)+\left(\frac{\hat{N}}{2}\log\left(\frac{m^2}{\mu^2}\right)\right)^2\right)+\ldots
\end{aligned}
$$

The Λ^2 divergences above have to be canceled by $\mathcal{A}_2^{(2)}$, which translates into the following identity:

$$
\begin{aligned}
&Cg_8\mu^{-2\varepsilon}\sum_i\alpha_i W_i^{(2)}\left(\frac{\hat{N}}{2}\log\left(\frac{m^2}{\mu^2}\right)\right)^2\\
&=-Cg_8\mu^{-2\varepsilon}\sum_i a^{(2)}_{2\,i}W_i^{(2)}\left(\frac{\hat{N}}{2}\log\left(\frac{m^2}{\mu^2}\right)\right)^2\\
&=-Cg_8\mu^{-2\varepsilon}\mathcal{A}_2^{(2)}\left(\frac{\hat{N}}{2}\log\left(\frac{m^2}{\mu^2}\right)\right)^2.
\end{aligned}
$$

In addition to the “genuine” double logs above, there are also contributions from one particle reducible topologies, the LSZ reduction, plus shifts of bare parameters to their renormalized values in lower order contributions of the process under consideration.

4 Equation of motion terms

In [15] it was shown that one has to allow for operators which vanish at the solution of the equation of motion (EOM terms in short) to define a basis in which one particle reducible (1PR) graphs contributing to the generating functional do not generate divergences. The equation of motion in euclidean space-time reads:

$$
\hat{X}:=\nabla_\mu u_\mu+\frac{\mathrm{i}}{2}\hat{\chi}_-=0,
\tag{4.1}
$$

with

$$
\begin{aligned}
\nabla_\mu u_\mu &:= \partial_\mu u_\mu+[\Gamma_\mu,u_\mu],\\
\Gamma_\mu &:= \frac{1}{2}\big(u^\dagger(\partial_\mu-\mathrm{i}r_\mu)+u(\partial_\mu-\mathrm{i}l_\mu)u^\dagger\big),\\
\hat{\chi}_- &:= \chi_- - \langle\chi_-\rangle/N.
\end{aligned}
$$

Before we can start with the actual calculation discussed in Sects. 5 and 6, we need to define this proper basis, relevant for our computation. Defining the generating functional of proper vertices:

$$
\bar{\Gamma}_a^{(n)}[J]=\Gamma_a^{(n)}[J,\phi]_{\phi=\bar{\phi}}:=Z_a^{(n)1\mathrm{PI}}[J];\ a=\mathrm{s},\mathrm{w},
\tag{4.2}
$$

evaluated at the solution of the EOM, and its functional derivative:

$$
\bar{\Gamma}_a^{(n)\,i}[J]=(\Gamma_a^{(n)\,i}[J,\phi])_{\phi=\bar{\phi}};\ a=\mathrm{s},\mathrm{w},
$$

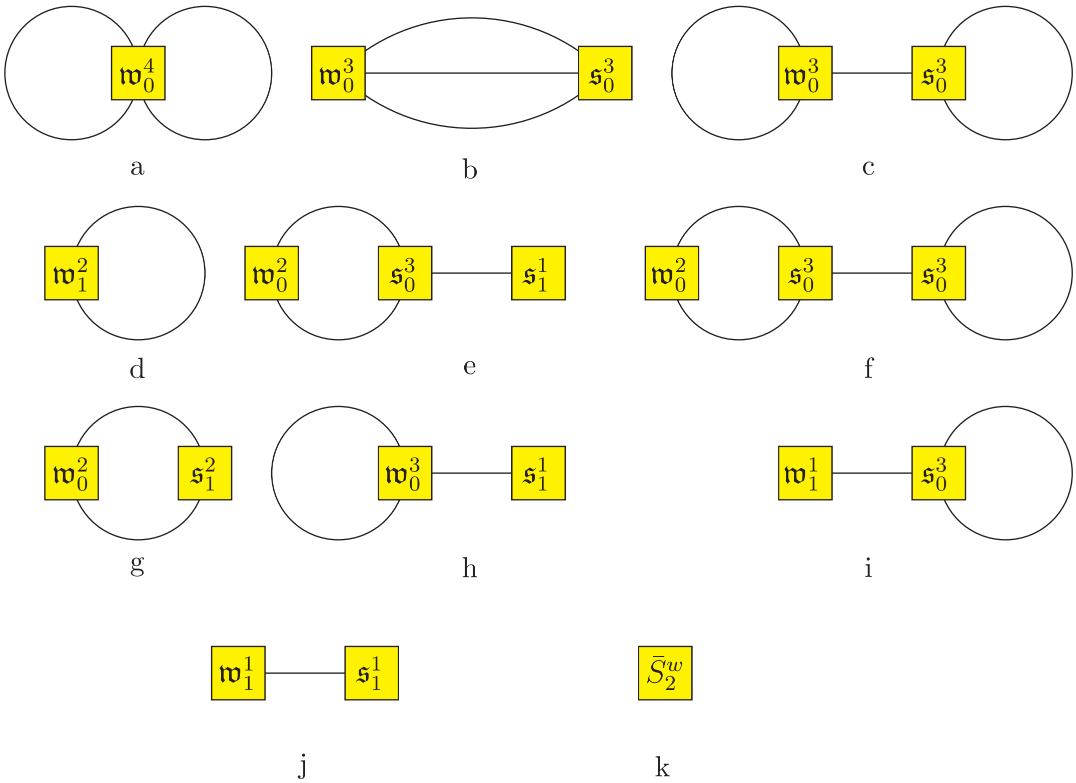

Fig. 1. Diagrams contributing to the generating functional $Z_{\mathrm{w}}^{(2)}$. They split into the class of 1PI diagrams (a,b,d,g), the 1PR diagrams (c,e,f,h,i,j), and the $\hbar$-order 2 action $\bar{S}_2^{\mathrm{w}}$, diagram k

we can decompose $Z_{\mathrm{w}}^{(2)}$ into a 1PI and 1PR part:

$$Z_{\mathrm{w}}^{(2)} = Z_{\mathrm{w}}^{(2)\mathrm{1PI}} + Z_{\mathrm{w}}^{(2)\mathrm{1PR}} = \bar{\Gamma}_{\mathrm{w}}^{(2)} - \bar{\Gamma}_{\mathrm{w}}^{(1)\,i} G_{ij} \bar{\Gamma}_{\mathrm{s}}^{(1)\,j}. \quad (4.3)$$

As (4.3) illustrates, the 1PR portion of Z will not contribute to divergences if in addition to Γ itself all its functional derivatives are finite. The latter can be achieved by appropriate additions of EOM terms to the Lagrangian [15]. (We will however conclude this section with a stronger proposition on this point.)

In the strong sector, we have at $\hbar$-order 1:

$$\hat{\mathcal{L}}_{\mathrm{s}}^{(1)} := \mathcal{L}_{\mathrm{GL}} + x_1^{(1)} \langle \chi_- \hat{X} \rangle + x_2^{(1)} \langle \hat{X}\hat{X} \rangle, \quad (4.4)$$

with $\mathcal{L}_{\mathrm{GL}}$ the usual Gasser–Leutwyler Lagrangian [16,17] and two additional EOM terms with coefficients $x_1^{(1)}, x_2^{(1)}$.

It was shown in [3] that the $\Gamma_{\mathrm{s}}^{(1)\,i}$ is finite if one discards the EOM terms altogether (i.e. $x_1^{(1)} = x_2^{(1)} = 0$), which is sufficient for our purposes. This result is to be expected since the only building block in $\mathcal{L}_{\mathrm{GL}}$ which corresponds to EOM terms, $\langle \chi_- \chi_- \rangle$, has a vanishing divergent counterterm.

In the weak sector, things get a little bit more involved. In addition to the EKW Lagrangian given in (3.6), we have six EOM terms:

$$\hat{\mathcal{L}}_{\mathrm{w}}^{(1)} = C g_8 F_0^2 \left(\sum_{i=1}^{37} c_i^{(1)} W_i^{(1)} + \sum_{i=1}^{6} e_i^{(1)} E_i^{(1)} \right), \quad (4.5)$$

Table 4. All EOM terms for $\mathcal{L}_{\mathrm{w}}^{(1)}$. In the last column we provide the operators of the EKW basis, given in Sect. 3, Table 3, which corresponds to the respective EOM term

i	$E_i^{(1)}$	k ($W_k^{(1)}$)
1	$\mathrm{i}\langle \Delta[u^2, \hat{X}] \rangle$	9
2	$\mathrm{i}\langle \Delta\{\chi_-, \hat{X}\} \rangle$	12
3	$\langle \Delta \hat{X}\hat{X} \rangle$	12
4	$\mathrm{i}\langle \Delta \hat{X} \rangle \langle \chi_- \rangle$	13
5	$\langle \Delta_\mu \{u_\mu, \hat{X}\} \rangle$	23
6	$\mathrm{i}\langle \Delta[\chi_+, \hat{X}] \rangle$	36

listed in Table 4.

In order to pin down the coefficients $e_i^{(1)}$, we calculate the functional derivative $\bar{\Gamma}_{\mathrm{w}}^{(1)\,i} = Z_{\mathrm{w}}^{(1)\,i}$ explicitly:

$$\bar{\Gamma}_{\mathrm{w}}^{(1)\,i} = \frac{1}{2} \bar{\mathfrak{w}}_0^{ijk} G_{jk} - \frac{1}{2} \bar{\mathfrak{s}}_0^{ijk} G_{jl} G_{km} \bar{\mathfrak{w}}_0^{lm} + \bar{\mathfrak{w}}_1^i. \quad (4.6)$$

The diagrammatic representation of (4.6) is shown in Fig. 2. To compute the ultraviolet divergent part of these diagrams, we have to expand the lowest order Lagrangians, (2.2) and (2.7), together with the counterterm Lagrangian $\mathcal{A}_1^1 = \sum Z_i W_i$, associated with (3.6). In Appendix Appendix E: we give the expansion of the building blocks defined in (D.1) in terms of the quantum fluctuations ξ and the background fields.

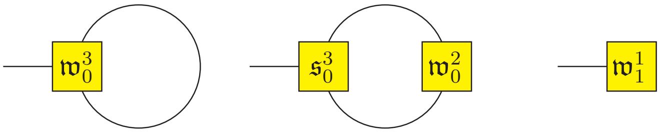

Fig. 2. Graphical representation of (4.6)

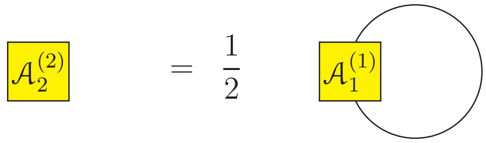

Fig. 3. Calculation of $\mathcal{A}_2^{(2)}$ with the RGE

The Wick contractions are then performed using the heat kernel representation of propagators, briefly explained in Appendix B. For the tadpole diagram $\bar{\mathfrak{w}}_0^{ijk} G_{jk}$, one uses the identities given in (B.8), whereas for the diagram $\bar{\mathfrak{s}}_0^{ijk} G_{jl} G_{km} \bar{\mathfrak{w}}_0^{lm}$, one employs the identities for products of propagators, provided in (B.5), (B.6) and (B.9). A more explicit discussion of how the computation works is given in Sect. 6.

The calculation, whose result is too lengthy to be displayed here, yields for the coefficients $e_i^{(1)}$ (euclidean space-time, N is the number of flavors):

$$
\begin{aligned}
e_1^{(1)} &= \frac{N}{2} \,; \\
e_2^{(1)} &= e_3^{(1)} = e_4^{(1)} = e_5^{(1)} = 0 \,; \\
e_6^{(1)} &= -\frac{N}{2} + \frac{2}{N}.
\end{aligned}
\tag{4.7}
$$

This shift of the original EKW Lagrangian corresponds to a replacement of the following building blocks:

$$
\begin{aligned}
W_9 = \langle \Delta[\chi_-, u^2] \rangle &\longrightarrow W_9' \cdot 2\mathrm{i} := \langle \Delta[\nabla_\mu u_\mu, u^2] \rangle \cdot 2\mathrm{i}, \\
W_{36} = \langle \Delta[\chi_+, \chi_-] \rangle &\longrightarrow W_{36}' \cdot 2\mathrm{i} := \langle \Delta[\chi_+, \nabla_\mu u_\mu] \rangle \cdot 2\mathrm{i}.
\end{aligned}
\tag{4.8}
$$

It is striking that W_9 and W_{36} are the only operators which can be shifted by EOM terms *and* have a non-vanishing counterterm coefficient $a_i^{(1)}$ (see Table 3). This observation leads to the conjecture that $\nabla_\mu u_\mu$ instead of $\hat{\chi}_-$ should be used in loop calculations. A more general discussion of this point will be given in a separate paper [18].

5 The calculation of $\mathcal{A}_2^{(2)}$

For the computation of $\mathcal{A}_2^{(2)}$, we use renormalization group techniques. If we write $\mathcal{L}_\mathrm{w}^{(2)}$, given in (3.10), split into the renormalized and counterterm part as follows:

$$
\mathcal{L}_\mathrm{w}^{(2)} = C g_8 \big(\mathcal{L}_\mathrm{w}^{(2)\,\mathrm{r}} + \mathcal{A}_1^{(2)} \Lambda + \mathcal{A}_2^{(2)} \Lambda^2 \big), \tag{5.1}
$$

the RGE imply the identity ($\partial_i^{(n)} = \partial / \partial c_i^{(n)\mathrm{r}}$):

$$
\mathcal{A}_2^{(2)} = \frac{1}{2} \boldsymbol{a}_1^{(1)} \boldsymbol{\partial}^{(1)} \mathcal{A}_1^{(2)}, \tag{5.2}
$$

with $\boldsymbol{a}_1^{(1)}$ being the counterterm of $\mathcal{L}_\mathrm{w}^{(1)}$, defined in (3.7). For the case of the weak non-leptonic sector, Fig. 1 shows all diagrams which can contribute to divergences at $\hbar$-order 2. If we act with the operator on the RHS of (5.2), $\boldsymbol{a}_1^{(1)} \boldsymbol{\partial}^{(1)}$, on all these diagrams and neglect the 1PR topologies (see Sect. 4), there is only diagram d which can give a non-vanishing contribution. The general diagrammatic representation of this statement, see (5.2), is shown in Fig. 3. A thorough discussion and derivation of this RGE approach can be found in [15]. The specific result, (5.2), was already used in [19].

Due to the RGE, we can therefore obtain the leading poles at $\hbar$-order 2 by computing only the one loop diagram d in Fig. 1 and weighting it with the factor 1/2, instead of having to calculate the genuine 1PI two loop diagrams a and b.

In Appendix C we provide all the operators of $\mathcal{A}_2^{(2)}$ which can contribute to amplitudes involving at most four pseudo-Goldstone particles and no vector sources, denoted by $\tilde{\mathcal{A}}_2^{(2)}$. This result has been used for the calculation of the double log contribution to the $K \to \pi\pi$ amplitude, which is presented in a separate paper [5]. The full expression is approximately four times the size of the truncated one shown in the appendix, and thus too lengthy to be displayed in this paper. It can however be obtained from the author. An overview about the workings of the actual computation can be found in Sect. 6.

6 Outline of the computation

In this section we provide a sketch of how the whole calculational machinery used in this paper works.

6.1 Tadpole graphs

We will illustrate the computation of a tadpole graph by computing a part of $\mathcal{A}_2^{(2)}$ with the help of (5.2). Our starting point is the counterterm of the NLO Lagrangian $\mathcal{L}_\mathrm{w}^{(1)}$ [1,2]. For completeness we provide the expanded building blocks needed in Appendix Appendix E:. We decided, however, not to reproduce the whole expanded expression of $\mathcal{A}_1^{(1)}$ here, since it is rather long.

We will restrict ourselves to the first building block which occurs in $\mathcal{A}_1^{(1)}$, $W_1 = \langle \Delta u^2 u^2 \rangle$, which we expand in the quantum fluctuation fields ξ:

$$
\begin{aligned}
&\langle \Delta u^2 u^2 \rangle \\
&\ = \langle \bar{\Delta} \bar{u}^2 \bar{u}^2 \rangle \\
&\quad + \frac{\mathrm{i}}{2} \langle \bar{\Delta} [\bar{u}^2 \bar{u}^2, \lambda_i] \rangle \xi^i - \langle \bar{\Delta} \{ \bar{u}^2, \{ \bar{u}_\mu, \lambda_i \} \} \rangle \xi_\mu^i
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{8}\langle\bar{\Delta}\big(\{\bar{u}^2, \{\bar{u}_\mu, [[\bar{u}_\mu, \lambda_i], \lambda_j]\}\} \\
&\qquad +[[\bar{u}^2\bar{u}^2, \lambda_i], \lambda_j]\big)\rangle\xi^i\xi^j \\
&- \frac{\mathrm{i}}{2}\langle\bar{\Delta}[\{\bar{u}^2, \{\bar{u}_\mu, \lambda_i\}\}, \lambda_j]\rangle\xi^i_\mu\xi^j \qquad (6.1)\\
&+ \langle\bar{\Delta}\big(\{\bar{u}^2, \lambda_i\lambda_j\} + \{\bar{u}_\mu, \lambda_i\}\{\bar{u}_\nu, \lambda_j\}\big)\rangle\xi^i_\mu\xi^j_\nu + \mathcal{O}(\xi^3).
\end{aligned}
$$

Due to delta functions generated by functional differentiation, no space-time integration survives and all building blocks and ξ fields in (6.1) are evaluated at the same space-time point x. Indices in $SU(N)$ space are denoted by i, j. The ξ fields of the bilinear terms proportional to $\xi^i\xi^j$, $\xi^i\xi^j_\mu$ and $\xi^i_\mu\xi^j_\nu$ are then contracted with the heat kernel representations of $G_\Delta(x,x)$, $d^x_\mu G_\Delta(x,y)|_{y=x}$ and $d^x_\mu d^y_\nu G_\Delta(x,y)|_{y=x}$ respectively, listed in (B.8). For the first contributing term in (6.1) we get the following contribution to the action $\bar{S}^{\mathrm{w}}_2$:

$$
\begin{aligned}
&-\frac{1}{8}\int \mathrm{d}^d x\langle\bar{\Delta}\big(\{\bar{u}^2, \{\bar{u}_\mu, [[\bar{u}_\mu, \lambda_i], \lambda_j]\}\} \\
&\qquad +[[\bar{u}^2\bar{u}^2, \lambda_i], \lambda_j]\big)\rangle G_\Delta(x,x)_{ij} \\
&= -\frac{(c\mu)^{-\varepsilon}\Lambda}{4}\int \mathrm{d}^d x\langle\bar{\Delta}\big(\{\bar{u}^2, \{\bar{u}_\mu, [[\bar{u}_\mu, \lambda_i], \lambda_j]\}\} \\
&\qquad +[[\bar{u}^2\bar{u}^2, \lambda_i], \lambda_j]\big)\rangle(a^\Delta_1)_{ij} + \text{finite terms} \\
&= -\frac{(c\mu)^{-\varepsilon}\Lambda}{4}\int \mathrm{d}^d x\langle\bar{\Delta}\big(\{\bar{u}^2, \{\bar{u}_\mu, [[\bar{u}_\mu, \lambda_i], \lambda_j]\}\} \\
&\qquad +[[\bar{u}^2\bar{u}^2, \lambda_i], \lambda_j]\big)\rangle \\
&- \frac{1}{8}\langle[u_\mu, \lambda_i][u_\mu, \lambda_j] + \{\lambda_i, \lambda_j\}\chi_+\rangle + \text{f.t.} \qquad (6.2)
\end{aligned}
$$

For the subsequent contraction of the $SU(N)$ indices i, j one uses the completeness relations:

$$
\begin{aligned}
&\sum_{i=1}^{N^2-1}\langle\lambda_i A\lambda_i B\rangle = -\frac{2}{N}\langle AB\rangle + 2\langle A\rangle\langle B\rangle, \\
&\sum_{i=1}^{N^2-1}\langle\lambda_i A\rangle\langle\lambda_i B\rangle = 2\langle AB\rangle - \frac{2}{N}\langle A\rangle\langle B\rangle. \qquad (6.3)
\end{aligned}
$$

This last step of the computation is obviously straightforward, and we forbear to display the final result, since it is again rather lengthy.

6.2 Beyond the tadpole

The computation of the functional derivative of $\Gamma^{(1)}_{\mathrm{w}}$, see (4.6), involves a diagram with two propagators and is therefore a little bit more involved: in addition to the $SU(N)$ contractions one has to deal with the space-time dependent part of the product of the two propagators. Let us again restrict ourselves to the simple case of a part of the computation where the vertices do not carry any derivatives acting on the ξ fields. The structure of such a piece is then:

$$
Q^a = \int \mathrm{d}^d x\mathrm{d}^d y\, v^{ajk}_{\mathrm{s}}(x)G_{jl}(x,y)G_{km}(x,y)v^{lm}_{\mathrm{w}}(y). \qquad (6.4)
$$

Here i, j, k, l, m are again pure $SU(N)$ indices, and a corresponds to the space-time point x_i as well as to the $SU(N)$ index i. The space-time dependent part can be evaluated with the help of (B.6) and (B.9) (we suppress the vertices $v^{ajk}_{\mathrm{s}}(x)$ and $v^{lm}_{\mathrm{w}}(y)$ in this step):

$$
\begin{aligned}
&\int \mathrm{d}^d x\mathrm{d}^d y G_{jl}(x,y)G_{km}(x,y) \\
&= \left(4\hat{N}\frac{\Gamma(1-\varepsilon/2)}{\pi^{-\varepsilon/2}}\right)^2 \\
&\times \int \mathrm{d}^d x\mathrm{d}^d y\, a^\Delta_0(x,y)_{jl}a^\Delta_0(x,y)_{km}|x-y|^{-d+\varepsilon} + \text{f.t.} \\
&= \left(4\hat{N}\frac{\Gamma(1-\varepsilon/2)}{\pi^{-\varepsilon/2}}\right)^2 (\pi)^{d/2}\frac{\Gamma(\varepsilon/2)}{\Gamma(d/2)} \\
&\times \int \mathrm{d}^d x\mathrm{d}^d y\, a^\Delta_0(x,y)_{jl}a^\Delta_0(x,y)_{km}\mu^{-\varepsilon}\delta^d(x-y) + \text{f.t.} \\
&= 2(c\mu)^{-\varepsilon}\Lambda\int \mathrm{d}^d x\, \delta_{jl}\delta_{km} + \text{f.t.} \qquad (6.5)
\end{aligned}
$$

Reinserting the vertices again, we finally get ($v^{ajk}(x) = \delta^d(x-x_i)v^{ijk}(\mathrm{x})$):

$$
\begin{aligned}
Q^a = Q^i(x_i) &= 2(c\mu)^{-\varepsilon}\Lambda\int \mathrm{d}^d x\, v^{ajk}_{\mathrm{s}}(x)v^{jk}_{\mathrm{w}}(x) + \text{f.t.} \\
&= 2(c\mu)^{-\varepsilon}\Lambda\, v^{ijk}_{\mathrm{s}}(x_i)v^{jk}_{\mathrm{w}}(x_i) + \text{f.t.} \qquad (6.6)
\end{aligned}
$$

If ξ fields with derivatives are contracted, one will in addition generate derivatives acting on the delta function in (6.5). These derivatives can be shifted to the vertices and Seeley–DeWitt coefficients by partial integration. As a last step, we will again have to contract the $SU(N)$ indices of the vertices with the ones of the Seeley–DeWitt coefficients of the expansion of the propagator, see (B.5), using once more the completeness relations in (6.3).

6.3 Verification of the calculation

This computation was exclusively performed with FORM 3.1 [20], a symbolic manipulation program. Since the whole calculation is thus fully automatized, one can conveniently adapt the code to problems whose solutions are known: In order to check the written code, we used it to recalculate two known counterterm Lagrangians. We replaced the original Lagrangian $\mathcal{A}^{(1)}_1$ (see Fig. 3) with the respective Lagrangians required for their calculation as starting point, but left the rest of the code unchanged.

(1) We recalculated the counterterms for $\mathcal{L}^{(1)}_{\mathrm{w}}$ using the method outlined in this paper, i.e. by the computation of the ring diagram $\frac{1}{2}\bar{\mathfrak{w}}^{ij}G_{ij}$ instead of using the logarithm: $\frac{1}{2}\mathrm{Tr}(\ln(\Delta_{\mathrm{s}} + \Delta_{\mathrm{w}}))$ and projecting out the part linear in G_{F}, as employed in the original calculation [1]. We found total agreement.

(2) We recalculated the leading poles at $\hbar$-order 2 in the strong sector and compared our result with the one obtained by Bijnens, Colangelo and Ecker [3,4]. The outcome of our computation matched completely with their result.

Since this is the very first NNLO calculation of CHPT in the weak sector, it was not possible to compare it directly with genuine two loop calculations. However, in our opinion, the two checks listed above, in particular the second one, are highly nontrivial, and yield sufficient evidence that the FORM code written for the computation is correct.

7 Conclusions

We have determined the leading divergences for the weak non-leptonic chiral Lagrangian at NNLO for the part which transforms like an octet under the chiral group, extending the analogue computation of the NLO counterterms [1,2]. The obtained result can be used to calculate double log contributions of observables, which at two loop order are the only quantities that do not depend on any LEC's [1]. Unlike in the strong sector, in the weak sector it is extremely difficult to determine these LEC's, and presumably one will not be able to pin them down in the near future, if ever. Thus, the double logs provide a first estimate about the NNLO corrections one has to expect, without the need of these unknown LEC's as input. Typically, the double log contributions in three flavor CHPT amount to around 10% of the corrections to the lowest order result.

Corrections to lowest order CHPT quantities are used for chiral extrapolations of lattice data. In these days, lattice simulations have entered a stage where one uses fully unquenched quarks, and aim to predict observables with an accuracy in the range of some percent. In view of such high precision, it certainly makes sense to include NNLO corrections in these extrapolations. The main objective of the present calculation is the use of the counterterm Lagrangian for the computation of the corresponding double log contributions to the $K \to \pi\pi$ amplitude in the $\Delta I = 1/2$ channel, which is presented in another paper [5]. Other interesting applications are analogue calculations for the $K \to \pi\pi\pi$ and $K \to \pi\gamma\gamma$ amplitudes.

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Appendix A: Background field method

In this appendix we provide a very brief outline of the background field formalism [21], which was used in the present calculation. The field ϕ in $U = \exp(\mathrm{i}\sqrt{2}\phi/F)$ is split into a background part $\bar{\phi}$, which is normally taken to be at the solution of the equation of motion, and a quantum fluctuation field ξ:

$$\phi = \bar{\phi} + \xi/\sqrt{2}. \tag{A.1}$$

[1] besides subtracted loop integrals which would require a fully-fledged two loop calculation.

$\bar{\phi}$ will be used as an external field, i.e. it will not propagate; therefore, only ξ can generate $\hbar$ corrections. With this substitution, the action assumes the form:

$$S[\phi] \to S[\bar{\phi}] + \frac{1}{2!}S[\bar{\phi}]^{ij}\xi_i\xi_j + \frac{1}{3!}S[\bar{\phi}]^{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}S[\bar{\phi}]^{ijkl}\xi_i\xi_j\xi_k\xi_l + \ldots \tag{A.2}$$

In the equation above we have assumed that $\bar{\phi}$ is a solution of the equation of motion, so that the linear term in the expansion vanishes. The RHS of (A.2) can now be viewed as the new Lagrangian where the new integration variables of the path integral are ξ instead of ϕ. The inverse of the bilinear operator in ξ, $S[\bar{\phi}]^{ij}(x,y) = \delta^d(x-y)\Delta_{ij}$, corresponds as usual to the propagator. Please note that we will only use the bilinear part of the strong chiral Lagrangian to define the propagator; the expanded form of the weak non-leptonic chiral Lagrangian will only be needed for the definition of vertices, each insertion thereof corresponding to a factor G_{F}. Δ is brought into the canonical form of an elliptical operator:

$$\Delta_{ij} = (-d_x^2 + \sigma(x))_{ij}, \tag{A.3}$$

$$d_{\mu\,kl} = \delta_{kl}\partial_\mu + \gamma_\mu(x)_{kl}, \tag{A.4}$$

which have the following explicit form for the CHPT Lagrangian, (2.2):

$$\gamma_{\mu\,ij} = -\frac{1}{2}\langle \Gamma_\mu[\lambda_i,\lambda_j]\rangle,$$
$$\sigma_{ij} = \frac{1}{8}\langle [u_\mu,\lambda_i][u_\mu,\lambda_j] + \{\lambda_i,\lambda_j\}\chi_+\rangle. \tag{A.5}$$

and will be used to define the propagator in Appendix B. The field strength associated to d_μ:

$$\gamma_{\mu\nu} = [d_\mu, d_\nu] = \partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu + [\gamma_\mu,\gamma_\nu], \tag{A.6}$$

will also be needed, and takes the explicit form:

$$\gamma_{\mu\nu\,ij} = -\frac{1}{2}\langle \Gamma_{\mu\nu}[\lambda_i,\lambda_j]\rangle; \quad \Gamma_{\mu\nu} = \frac{1}{4}[u_\mu,u_\nu] - \frac{\mathrm{i}}{2}f_{+\mu\nu}. \tag{A.7}$$

Appendix B: The heat kernel method

In this appendix we give a brief summary of the heat kernel method as developed by Jack and Osborn [22] and provide a compilation of the results which are needed for the calculation. Throughout this section we work in euclidean space-time. The presentation is to a large extent based on [22,4] and some results provided in [23].

Let us consider the propagator $G_\Delta(x,y)$ associated to Δ, which in euclidean space-time is defined by:

$$\Delta_x G_\Delta(x,y) = \delta(x-y). \tag{B.1}$$

In the Schwinger representation, $G_\Delta(x,y)$ is written as an integral over the eigentime τ:

$$G_\Delta(x,y) = \int_0^\infty \mathrm{d}\tau \rho(\tau,\varepsilon)\mathcal{G}_\Delta(x,y;\tau). \tag{B.2}$$

The kernel of the integral, $\mathcal{G}_\Delta(x,y;\tau)$, satisfies the diffusion equation:

$$\partial_\tau \mathcal{G}_\Delta(x,y;\tau) = -\Delta_x \mathcal{G}_\Delta(x,y;\tau) \tag{B.3}$$

and the boundary condition $\mathcal{G}_\Delta(x,y;0) = \delta(x-y)$.

$\mathcal{G}_\Delta(x,y;\tau)$ can be expanded in the Seeley–DeWitt coefficients:

$$\mathcal{G}_\Delta(x,y;\tau) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \mathrm{e}^{\frac{|x-y|^2}{4\tau}} \sum_{j=0}^\infty a_j^\Delta(x,y)\tau^j, \tag{B.4}$$

and after solving the integral (B.2) with the regulator function $\rho(\varepsilon,\tau) = (4\pi\tau)^{-\frac{\varepsilon}{2}}$ corresponding to dimensional regularization, one gets the asymptotic formula:

$$\begin{aligned} &G_\Delta(x,y) \\ &= G_0(x-y)a_0^\Delta(x,y) + R_1(x-y;c\mu)a_1^\Delta(x,y) \\ &\quad +R_2(x-y;c\mu)a_2^\Delta(x,y) \\ &\quad +R_3(x-y;c\mu)a_3^\Delta(x,y) + \overline{G}_\Delta(x,y;c\mu), \end{aligned} \tag{B.5}$$

with the coefficients:

$$\begin{aligned} G_0(x) &= \hat{N}\frac{\Gamma\left(1-\frac{\varepsilon}{2}\right)}{4\pi^{-\frac{\varepsilon}{2}}}|x|^{\varepsilon-2}, \\ R_1(x;c\mu) &= 2(c\mu)^{-\varepsilon}\Lambda + \hat{N}\frac{\Gamma\left(-\frac{\varepsilon}{2}\right)}{\pi^{\frac{\varepsilon}{2}}}|x|^\varepsilon, \\ R_2(x;c\mu) &= \frac{|x|^2}{4}\left(-2(c\mu)^{-\varepsilon}\Lambda + \hat{N}\frac{\Gamma\left(-1-\frac{\varepsilon}{2}\right)}{\pi^{\frac{\varepsilon}{2}}}|x|^\varepsilon\right), \\ R_3(x;c\mu) &= |x|^4\left((c\mu)^{-\varepsilon}\Lambda + \hat{N}\frac{\Gamma\left(-2-\frac{\varepsilon}{2}\right)}{\pi^{\frac{\varepsilon}{2}}}|x|^\varepsilon\right), \end{aligned} \tag{B.6}$$

where c parametrizes the renormalization prescription ($\ln(c) = \exp(-(\ln(4\pi) + \Gamma'(1) + 1)/2$ for $\overline{\mathrm{MS}}$.).

Equation (B.5) is only valid asymptotically for $\tau \to 0$ and can therefore only be used to extract the ultraviolet behavior of the propagator. The Seeley–DeWitt coefficients are given by:

$$\begin{aligned} a_0^\Delta &= 1, \\ a_1^\Delta &= -\sigma, \\ a_2^\Delta &= \frac{1}{12}\gamma_{\mu\nu}\gamma_{\mu\nu} + \frac{1}{2}\sigma^2 - \frac{1}{6}d_\mu d_\mu \sigma, \end{aligned} \tag{B.7}$$

with σ and the field strength $\gamma_{\mu\nu}$ defined in (A.5) and (A.6) respectively.

In order to calculate the divergences of the tadpole graphs like Fig. 3, we need the propagator with up to three derivatives, which can be calculated by use of (B.6), projecting out the space-time independent part:

$$G_\Delta(x,x) = (c\mu)^{-\varepsilon}\Lambda\, 2a_1^\Delta(x,x) + \overline{G}_\Delta(x,x;c\mu),$$

$$\begin{aligned} &d_\mu^x G_\Delta(x,y)\big|_{y=x} \\ &= (c\mu)^{-\varepsilon}\Lambda\, 2d_\mu^x a_1^\Delta(x,y)\big|_{y=x} \\ &+ d_\mu^x \overline{G}_\Delta(x,y;c\mu)\big|_{y=x}, \\ &d_\mu^x d_\nu^y G_\Delta(x,y)\big|_{y=x} \\ &= (c\mu)^{-\varepsilon}\Lambda\Big(2d_\mu^x d_\nu^y a_1^\Delta(x,y)\Big|_{y=x} + \delta_{\mu\nu}a_2^\Delta(x,x)\Big) \\ &+ d_\mu^x d_\nu^y \overline{G}_\Delta(x,y;c\mu)\big|_{y=x}, \\ &d_\mu^x d_\nu^x G_\Delta(x,y)\big|_{y=x} \\ &= (c\mu)^{-\varepsilon}\Lambda\Big(2d_\mu^x d_\nu^x a_1^\Delta(x,y)\Big|_{y=x} - \delta_{\mu\nu}a_2^\Delta(x,x)\Big) \\ &+ d_\mu^x d_\nu^x \overline{G}_\Delta(x,y;c\mu)\big|_{y=x}, \\ &d_\mu^x d_\nu^x d_\rho^y G_\Delta(x,y)\big|_{y=x} \\ &= (c\mu)^{-\varepsilon}\Lambda\Big(2d_\mu^x d_\nu^x d_\rho^y a_1^\Delta(x,x) \\ &\quad -\big(\delta_{\mu\nu}d_\rho^y a_2^\Delta(x,x) - \delta_{\mu\rho}d_\nu^x a_2^\Delta(x,x) \\ &\qquad -\delta_{\nu\rho}d_\mu^x a_2^\Delta(x,x)\big)\Big) \\ &+ d_\mu^x d_\nu^x d_\rho^y \overline{G}_\Delta(x,y;c\mu)\big|_{y=x}. \end{aligned} \tag{B.8}$$

For the calculation of the divergences of the functional derivative of $Z_{\mathrm{w}}^{(1)}$, see (4.6) or Fig. 2, one needs to deal with products of propagators. After a couple of manipulations, the space-time dependent part of such products can be brought into the form of a sum of terms proportional to:

$$\prod_{j=1}^n \partial_{\alpha_j}^z \frac{1}{|z|^{2m}}; \quad z := y - x; \quad \partial_\mu^x = -\partial_\mu^z; \quad \partial_\mu^y = \partial_\mu^z.$$

Such terms can be represented by delta functions via their Fourier transforms in d space-time dimensions:

$$\int \mathrm{d}^d z \frac{1}{|z|^{2m}} = \pi^{\frac{d}{2}}\frac{\Gamma(-(m-d/2))}{\Gamma(m)}\left(q^2\right)^{m-d/2}, \tag{B.9}$$

substituting $q^{2n-k\varepsilon} \to \mu^{-k\varepsilon}(-\partial^2)^n\delta^d(z)$; $n,k \in \mathbb{N}_0$, from which the ultraviolet divergent parts can easily be extracted.

Appendix C: Explicit result for $\mathcal{A}_2^{(2)}$

We have:

$$\begin{aligned} &\tilde{\mathcal{A}}_2^2 \\ &= \langle\Delta\chi_+\chi_+\chi_+\rangle\left(-\frac{21}{8} + \frac{6}{N^2} - \frac{N}{8} + \frac{5N^2}{32}\right) \\ &+ \langle\Delta\chi_+\chi_+\rangle\langle\chi_+\rangle\left(\frac{1}{2} - \frac{5}{N^3} - \frac{1}{N} + \frac{N}{2} - \frac{N^2}{32}\right) \\ &+ \langle\Delta\chi_+\rangle\langle\chi_+\chi_+\rangle\left(\frac{1}{4} - \frac{2}{N^3} + \frac{3}{8N} + \frac{5N}{32} - \frac{N^2}{32}\right) \\ &+ \langle\Delta\chi_-\rangle\langle\chi_+\chi_-\rangle\left(-\frac{2}{N^3} - \frac{5}{4N} + \frac{N}{16}\right) \end{aligned}$$

$$+ \langle \Delta\chi_- \rangle\langle \chi_+ \rangle\langle \chi_- \rangle \left(\frac{2}{N^4} + \frac{1}{2N^2} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle \chi_+ \rangle\langle \chi_+ \rangle \left(\frac{1}{8} + \frac{2}{N^4} + \frac{9N^2}{8} - \frac{7}{16N} - \frac{3N}{32} \right)$$
$$+ \langle \Delta\chi_-\chi_- \rangle\langle \chi_+ \rangle \left(-\frac{3}{N^3} - \frac{3}{8N} + \frac{3N}{32} \right)$$
$$+ \langle \Delta\chi_-\chi_+\chi_- \rangle \left(-\frac{3}{8} + \frac{4}{N^2} + \frac{N^2}{32} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle \chi_-\chi_- \rangle \left(-\frac{1}{8N} - \frac{N}{32} \right)$$
$$+ \langle \{\Delta\chi_-\chi_-\chi_+\} \rangle \left(-\frac{1}{8} + \frac{2}{N^2} + \frac{N^2}{32} \right)$$
$$+ \langle \{\Delta\chi_-\chi_+\} \rangle\langle \chi_- \rangle \left(-\frac{1}{N^3} - \frac{5}{8N} + \frac{N}{32} \right)$$
$$+ \langle [\Delta\chi_-\chi_+] \rangle\langle \chi_+ \rangle \left(\frac{1}{2N^3} + \frac{5}{8N} - \frac{N}{16} \right)$$
$$+ \langle [\Delta\chi_-\chi_+\chi_+] \rangle \left(-\frac{1}{8} - \frac{1}{2N^2} \right)$$
$$+ \langle \Delta{\chi_+}_\mu^\mu \rangle\langle \chi_+ \rangle \left(\frac{3}{8} - \frac{1}{2N} + \frac{N}{16} - \frac{N^2}{16} \right)$$
$$+ \langle [\Delta\chi_-{\chi_+}_\mu^\mu] \rangle \left(-\frac{1}{4} + \frac{N^2}{16} \right)$$
$$+ \langle \Delta u_\mu\chi_+u^\mu \rangle\langle \chi_+ \rangle \left(-\frac{9}{16} + \frac{7}{8N} - \frac{3N}{32} + \frac{3N^2}{32} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle {\chi_+}_\mu^\mu \rangle \left(\frac{1}{16} + \frac{7}{8N} - \frac{N}{8} - \frac{N^2}{16} \right)$$
$$+ \langle \Delta u_\mu \rangle\langle \chi_+u^\mu \rangle\langle \chi_+ \rangle \left(\frac{1}{8} + \frac{5}{4N^2} + \frac{1}{8N} + \frac{N}{8} \right)$$
$$+ \langle \Delta u_\mu \rangle\langle \chi_+\chi_+u^\mu \rangle \left(-\frac{1}{4} - \frac{2}{N^1} + \frac{N}{32} + \frac{N^2}{16} \right)$$
$$+ \langle \{\Delta{\chi_+}_\mu^\mu\chi_+\} \rangle \left(-\frac{3}{8} - \frac{N}{16} + \frac{N^2}{8} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle \chi_+u^2 \rangle \left(-\frac{1}{16} - \frac{15}{8N} + \frac{3N}{4} + \frac{3N^2}{32} \right)$$
$$+ \langle \Delta u_\mu\chi_+\chi_+u^\mu \rangle \left(-\frac{1}{4} + \frac{N}{16} \right)$$
$$+ \langle \Delta u^2 \rangle\langle \chi_+ \rangle\langle \chi_+ \rangle \left(-\frac{3}{32} - \frac{7}{8N^2} + \frac{1}{8N} + \frac{3N}{32} \right)$$
$$+ \langle \Delta u^2 \rangle\langle \chi_+\chi_+ \rangle \left(-\frac{1}{4} + \frac{9}{8N} - \frac{N}{16} + \frac{N^2}{16} \right)$$
$$+ \langle \Delta u^2\chi_+ \rangle\langle \chi_+ \rangle \left(-\frac{3}{32} + \frac{1}{8N} - \frac{N}{64} + \frac{N^2}{64} \right)$$
$$+ \langle \Delta\chi_+u^2 \rangle\langle \chi_+ \rangle \left(-\frac{3}{32} + \frac{1}{8N} - \frac{N}{64} + \frac{N^2}{64} \right)$$
$$+ \langle \Delta\chi_+u^2\chi_+ \rangle \left(-\frac{1}{2} - \frac{3N^2}{32} \right)$$
$$+ \langle \Delta\chi_+\chi_+ \rangle\langle u^2 \rangle \left(-\frac{1}{8} - \frac{19}{8N} + \frac{17N}{32} + \frac{N^2}{32} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle \chi_+ \rangle\langle u^2 \rangle \left(\frac{11}{32} + \frac{17}{8N^2} + \frac{1}{16N} - \frac{3N}{32} \right)$$
$$+ \langle \{\Delta u_\mu\chi_+\} \rangle\langle \chi_+u^\mu \rangle \left(\frac{3}{4N} - \frac{3N}{32} + \frac{N^2}{32} \right)$$
$$+ \langle \{\Delta u^2\chi_+\chi_+\} \rangle \left(-\frac{7}{16} - \frac{N}{16} - \frac{5N^2}{64} \right)$$
$$+ \langle \{\Delta u_\mu\chi_+u^\mu\chi_+\} \rangle \left(\frac{1}{2} + \frac{3N}{32} - \frac{3N^2}{16} \right)$$
$$+ \langle \{\Delta u^2\chi_+\} \rangle\langle \chi_+ \rangle \left(\frac{3}{16} + \frac{1}{16N} - \frac{N}{64} + \frac{N^2}{64} \right)$$
$$+ \mathrm{i}\langle [\Delta\chi_{+\mu}u^\mu\chi_+] \rangle \left(-\frac{9}{16} - \frac{N}{32} + \frac{N^2}{8} \right)$$
$$+ \mathrm{i}\langle [\Delta u_\mu{\chi_+}^\mu] \rangle\langle \chi_+ \rangle \left(-\frac{3}{16} + \frac{1}{4N} - \frac{N}{8} + \frac{N^2}{32} \right)$$
$$+ \mathrm{i}\langle [\Delta u_\mu{\chi_+}^\mu\chi_+] \rangle \left(-\frac{1}{4} + \frac{N^2}{32} \right)$$
$$+ \mathrm{i}\langle [\Delta u_\mu\chi_+] \rangle\langle {\chi_+}^\mu \rangle \left(-\frac{3}{4N} + \frac{5N}{32} \right)$$
$$+ \mathrm{i}\langle [\Delta u_\mu\chi_+{\chi_+}^\mu] \rangle \left(\frac{5}{16} + \frac{N}{32} \right)$$
$$+ \langle \Delta h_{\mu\nu}h^{\mu\nu} \rangle\langle \chi_+ \rangle \left(\frac{N}{32} - \frac{N^2}{32} \right)$$
$$+ \langle \Delta h_{\mu\nu}\chi_+h^{\mu\nu} \rangle \left(-\frac{1}{8} + \frac{N}{16} \right)$$
$$+ \langle \Delta\chi_+ \rangle\langle h_{\mu\nu}h^{\mu\nu} \rangle \left(-\frac{N}{16} - \frac{N^2}{32} \right)$$
$$+ \langle \{\Delta\chi_+h_{\mu\nu}h^{\mu\nu}\} \rangle \left(\frac{1}{16} - \frac{N}{32} + \frac{N^2}{16} \right)$$
$$+ \langle [\Delta\chi_-h_{\mu\nu}h^{\mu\nu}] \rangle \left(\frac{N^2}{32} \right)$$
$$+ \langle [\Delta u_\mu\chi_-u^\mu\chi_+] \rangle \left(\frac{9}{32} + \frac{N}{64} - \frac{N^2}{16} \right)$$
$$+ \langle [\Delta u_\mu\chi_+u^\mu\chi_-] \rangle \left(\frac{N^2}{64} \right)$$
$$+ \langle \Delta u_\mu \rangle\langle [\chi_+u^\mu\chi_-] \rangle \left(-\frac{1}{4N} - \frac{3N}{32} \right)$$
$$+ \langle [\Delta\chi_-u^2\chi_+] \rangle \left(\frac{9}{32} + \frac{N}{64} - \frac{5N^2}{64} \right)$$
$$+ \langle [\Delta u_\mu\chi_-] \rangle\langle \chi_+u^\mu \rangle \left(\frac{3}{4N} - \frac{N}{96} - \frac{N^2}{96} \right)$$
$$+ \langle [\Delta u_\mu\chi_-u^\mu] \rangle\langle \chi_+ \rangle \left(\frac{3}{32} - \frac{1}{8N} + \frac{N}{16} - \frac{N^2}{64} \right)$$
$$+ \langle [\Delta u_\mu\chi_-u^\mu\chi_+] \rangle \left(-\frac{1}{32} - \frac{1}{N^2} - \frac{3N}{64} - \frac{N^2}{32} \right)$$
$$+ \langle [\Delta u_\mu\chi_-\chi_+u^\mu] \rangle \left(-\frac{5}{96} + \frac{5N}{192} \right)$$
$$+ \langle [\Delta u_\mu\chi_+] \rangle\langle \chi_-u^\mu \rangle \left(\frac{1}{8} + \frac{7}{4N} - \frac{11N}{32} \right)$$

$$
\begin{aligned}
&+ \langle[\Delta u_\mu \chi_+ u^\mu \chi_-]\rangle \left(-\frac{5}{32} - \frac{N}{64} + \frac{N^2}{64}\right) \\
&+ \langle[\Delta u_\mu \chi_+ \chi_- u^\mu]\rangle \left(-\frac{5}{32} - \frac{N}{64}\right) \\
&+ \langle[\Delta u^2 \chi_-]\rangle\langle\chi_+\rangle \left(\frac{3}{32} - \frac{1}{4N} + \frac{29N}{96} - \frac{N^2}{48}\right) \\
&+ \langle[\Delta u^2 \chi_- \chi_+]\rangle \left(-\frac{7}{48} + \frac{1}{2N^2} - \frac{N}{12} + \frac{3N^2}{32}\right) \\
&+ \langle[\Delta \chi_- u^2 \chi_+]\rangle \left(-\frac{19}{96} + \frac{1}{2N^2} + \frac{N}{192} - \frac{N^2}{64}\right) \\
&+ \langle[\Delta u^2 \chi_+ \chi_-]\rangle \left(\frac{N^2}{64}\right) \\
&+ \langle[\Delta \chi_- u_\mu \chi_+ u^\mu]\rangle \left(\frac{29}{96} + \frac{7N}{192} - \frac{N^2}{16}\right) \\
&+ \langle[\Delta \chi_- \chi_+]\rangle\langle u^2\rangle \left(\frac{1}{16} + \frac{1}{4N} - \frac{N^2}{64}\right) \\
&+ \langle[\Delta \chi_- \chi_+ u^2]\rangle \left(\frac{N}{16} - \frac{3N^2}{64}\right) \\
&+ \langle[\Delta u^2 \chi_+]\rangle\langle\chi_-\rangle \left(\frac{1}{8} - \frac{3}{4N} + \frac{N}{8}\right) \\
&+ \mathrm{i}\langle\Delta u_\mu\rangle\langle{\chi_-}^\mu\rangle\langle\chi_+\rangle \left(-\frac{1}{16} + \frac{N}{16}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu {\chi_-}^\mu\}\rangle\langle\chi_+\rangle \left(\frac{3}{8} - \frac{1}{2N} + \frac{N}{16} - \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu {\chi_-}^\mu \chi_+\}\rangle \left(-\frac{3}{8} - \frac{N}{16} + \frac{N^2}{8}\right) \\
&+ \mathrm{i}\langle\{\Delta \chi_{-\mu} u^\mu \chi_+\}\rangle \left(-\frac{3}{8} - \frac{N}{16} + \frac{N^2}{8}\right) \\
&+ \mathrm{i}\langle\Delta u_\mu\rangle\langle[{h^\mu}_\nu u^\nu \chi_+]\rangle \left(\frac{N}{8}\right) \\
&+ \mathrm{i}\langle\Delta \chi_+\rangle\langle\chi_{+\mu} u^\mu\rangle \left(\frac{1}{8} + \frac{7}{4N} - \frac{3N}{8} - \frac{3N^2}{16}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu {\chi_-}^\mu\}\rangle\langle\chi_+\rangle \left(\frac{N}{32} - \frac{N^2}{32}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu \chi_+\}\rangle\langle{\chi_-}^\mu\rangle \left(-\frac{N}{8}\right) \\
&+ \mathrm{i}\langle\{\Delta \chi_{-\mu} u^\mu \chi_+\}\rangle \left(\frac{1}{16} - \frac{N}{32} + \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu {\chi_-}^\mu \chi_+\}\rangle \left(\frac{1}{16} - \frac{N}{32} + \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle[\Delta u_\mu u_\nu h^{\mu\nu}]\rangle\langle\chi_+\rangle \left(-\frac{N}{24} + \frac{N^2}{96}\right) \\
&+ \mathrm{i}\langle[\Delta u_\mu u_\nu \chi_+ h^{\mu\nu}]\rangle \left(\frac{1}{16} - \frac{N}{32}\right) \\
&+ \mathrm{i}\langle[\Delta u_\mu {h^\mu}_\nu]\rangle\langle\chi_+ u^\nu\rangle \left(\frac{7N}{48} - \frac{N^2}{96}\right) \\
&+ \mathrm{i}\langle\{\Delta \chi_{-\mu} \chi_+ u^\mu\}\rangle \left(-\frac{1}{8} + \frac{N}{16}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathrm{i}\langle[\Delta u_\mu \chi_+]\rangle\langle{h^\mu}_\nu u^\nu\rangle \left(-\frac{1}{N} + \frac{N}{4}\right) \\
&+ \mathrm{i}\langle[\Delta u_\mu \chi_+ u_\nu h^{\mu\nu}]\rangle \left(-\frac{1}{48} + \frac{N}{96}\right) \\
&+ \mathrm{i}\langle[\Delta \chi_+ u_\mu u_\nu h^{\mu\nu}]\rangle \left(\frac{1}{24} - \frac{N}{48} - \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle[\Delta u_\mu \chi_+ {h^\mu}_\nu u^\nu]\rangle \left(-\frac{5}{48} + \frac{5N}{96}\right) \\
&+ \mathrm{i}\langle[\Delta \chi_+ u_\mu {h^\mu}_\nu u^\nu]\rangle \left(\frac{7}{16} + \frac{N}{32} - \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle[\Delta \chi_+ h_{\mu\nu}]\rangle\langle u^\mu u^\nu\rangle \left(\frac{1}{2N} - \frac{3N}{32}\right) \\
&+ \mathrm{i}\langle[\Delta \chi_+ h_{\mu\nu} u^\mu u^\nu]\rangle \left(\frac{1}{12} - \frac{N}{24} - \frac{N^2}{32}\right) \\
&+ \mathrm{i}\langle\Delta u_\mu\rangle\langle{\chi_+}^\mu\rangle\langle\chi_-\rangle \left(\frac{1}{8} + \frac{1}{2N^2} - \frac{3}{8N} - \frac{N}{16}\right) \\
&+ \mathrm{i}\langle\Delta u_\mu\rangle\langle{\chi_+}^\mu \chi_-\rangle \left(\frac{1}{2} - \frac{3}{4N} + \frac{5N}{16} - \frac{N^2}{16}\right) \\
&+ \mathrm{i}\langle\Delta \chi_{+\mu}\rangle\langle\chi_- u^\mu\rangle \left(\frac{1}{8} - \frac{1}{4N} + \frac{3N}{16}\right) \\
&+ \mathrm{i}\langle\Delta \chi_-\rangle\langle\chi_{+\mu} u^\mu\rangle \left(\frac{1}{8} + \frac{3}{4N} + \frac{N}{16}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu \chi_-\}\rangle\langle{\chi_+}^\mu\rangle \left(\frac{1}{2N} + \frac{N}{32}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu \chi_- {\chi_+}^\mu\}\rangle \left(-\frac{1}{4} - \frac{3N}{32}\right) \\
&+ \mathrm{i}\langle\{\Delta \chi_- u_\mu {\chi_+}^\mu\}\rangle \left(-\frac{1}{4} - \frac{N}{16} - \frac{N^2}{32}\right) \\
&+ \mathrm{i}\langle\{\Delta \chi_- \chi_{+\mu} u^\mu\}\rangle \left(-\frac{5}{8} - \frac{3N}{32} + \frac{N^2}{16}\right) \\
&+ \langle\Delta u_\mu\rangle\langle{\chi_+}^{\{\mu}{}_{\nu\}} u^\nu\rangle \left(-\frac{1}{3N} + \frac{N}{8}\right) \\
&+ \langle\Delta u_\mu\rangle\langle{\chi_+}^\nu{}_\nu u^\mu\rangle \left(-\frac{4}{3N} + \frac{3N}{8}\right) \\
&+ \langle\Delta u_\mu u_\nu\rangle\langle{\chi_+}^{\{\mu\nu\}}\rangle \left(\frac{1}{6N} + \frac{N}{72}\right) \\
&+ \langle\Delta u_\mu {\chi_+}^\nu{}_\nu u^\mu\rangle \left(-\frac{1}{36} - \frac{N^2}{72}\right) \\
&+ \langle\Delta {\chi_+}^\mu{}_\mu\rangle\langle u^2\rangle \left(-\frac{2}{3N} - \frac{N}{72}\right) \\
&+ \mathrm{i}\langle\{\Delta u_\mu {\chi_+}^\mu\}\rangle\langle\chi_-\rangle \left(\frac{1}{8} - \frac{1}{8N} + \frac{3N}{32}\right) \\
&+ \langle\Delta \chi_{+\mu\nu}\rangle\langle u^\mu u^\nu\rangle \left(-\frac{1}{3N} + \frac{N}{18}\right) \\
&+ \langle\Delta u^2\rangle\langle{\chi_+}^\mu{}_\mu\rangle \left(\frac{2}{3N} + \frac{11N}{36}\right) \\
&+ \langle\{\Delta u_\mu u_\nu {\chi_+}^{\{\mu\nu\}}\}\rangle \left(\frac{1}{72} - \frac{N^2}{144}\right)
\end{aligned}
$$

$$+ \langle \Delta\chi_-\rangle\langle\chi_- u^2\rangle \left(-\frac{1}{8} - \frac{3}{2N}\right)$$
$$+ \langle\{\Delta u^2 \chi_{+\mu}{}^{\mu}\}\rangle \left(-\frac{23}{72} + \frac{23N^2}{144}\right)$$
$$+ \mathrm{i}\langle[\Delta\chi_- u_\mu \chi_-{}^{\mu}]\rangle \left(\frac{N^2}{32}\right)$$
$$+ \langle\{\Delta u_\mu \chi_+{}^{\mu}{}_{\nu} u^\nu\}\rangle \left(-\frac{7}{36} + \frac{N^2}{36}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu \chi_-{}^{\mu} \chi_-]\rangle \left(-\frac{N^2}{32}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu \chi_-]\rangle\langle\chi_-{}^{\mu}\rangle \left(\frac{N}{16}\right)$$
$$+ \mathrm{i}\langle[\Delta\chi_- u_\mu \chi_-{}^{\mu}]\rangle \left(-\frac{1}{4} + \frac{N^2}{16}\right)$$
$$+ \mathrm{i}\langle[\Delta\chi_- \chi_{+\mu} u^\mu]\rangle \left(-\frac{1}{4} + \frac{N^2}{16}\right)$$
$$+ \langle\Delta u_\mu\rangle\langle\chi_- u^\mu\rangle\langle\chi_-\rangle \left(-\frac{1}{8} - \frac{1}{2N^2} + \frac{3}{8N} + \frac{N}{16}\right)$$
$$+ \langle\Delta u_\mu\rangle\langle\chi_-\chi_- u^\mu\rangle \left(-\frac{1}{2} + \frac{1}{N} - \frac{N}{4} + \frac{N^2}{16}\right)$$
$$+ \langle\Delta u_\mu \chi_-\rangle\langle\chi_- u^\mu\rangle \left(-\frac{1}{16} + \frac{1}{8N} - \frac{3N}{32}\right)$$
$$+ \langle\Delta u^2\rangle\langle\chi_-\chi_-\rangle \left(-\frac{1}{8N} - \frac{N}{32}\right)$$
$$+ \langle\Delta\chi_-\chi_-\rangle\langle u^2\rangle \left(-\frac{5}{8N} + \frac{3N}{32}\right)$$
$$+ \langle\Delta\chi_- u_\mu\rangle\langle\chi_- u^\mu\rangle \left(-\frac{1}{16} + \frac{1}{8N} - \frac{3N}{32}\right)$$
$$+ \langle\Delta\chi_- u^2 \chi_-\rangle \left(\frac{5}{24} + \frac{1}{N^2} + \frac{N}{32} + \frac{7N^2}{96}\right)$$
$$+ \langle\{\Delta u_\mu \chi_-\}\rangle\langle\chi_- u^\mu\rangle \left(\frac{1}{4N} - \frac{3N}{32}\right)$$
$$+ \langle\Delta u_\mu \chi_- u^\mu\rangle\langle\chi_-\rangle \left(-\frac{1}{8} + \frac{1}{8N} - \frac{N}{16}\right)$$
$$+ \langle\{\Delta u_\mu \chi_- u^\mu \chi_-\}\rangle \left(\frac{13}{48} - \frac{1}{N^2} + \frac{3N}{32} - \frac{N^2}{96}\right)$$
$$+ \langle\Delta u_\mu \chi_-\chi_- u^\mu\rangle \left(\frac{5}{24} + \frac{N}{8}\right)$$
$$+ \langle\{\Delta u^2 \chi_-\}\rangle\langle\chi_-\rangle \left(-\frac{1}{16} - \frac{5}{16N} - \frac{N}{32}\right)$$
$$+ \langle\{\Delta u^2 \chi_-\chi_-\}\rangle \left(\frac{1}{3} + \frac{1}{2N^2} + \frac{N}{16} - \frac{N^2}{96}\right)$$
$$+ \langle\Delta u_\mu\rangle\langle h_{\nu\rho} h^{\nu\rho} u^\mu\rangle \left(\frac{3N}{16}\right)$$
$$+ \langle\Delta u_\mu\rangle\langle\{h^\mu{}_\nu h^\nu{}_\rho u^\rho\}\rangle \left(\frac{N}{16}\right)$$
$$+ \langle\Delta u_\mu u_\nu\rangle\langle h^\mu{}_\rho h^{\nu\rho}\rangle \left(\frac{N}{36}\right) + \langle\Delta u_\mu u_\nu\rangle\langle h^\nu{}_\rho h^{\mu\rho}\rangle \left(-\frac{N}{72}\right)$$

$$+ \langle\Delta u_\mu h^\mu{}_\nu h^\nu{}_\rho u^\rho\rangle \left(-\frac{N^2}{36}\right) + \langle\Delta u_\mu h_{\nu\rho} h^{\nu\rho} u^\mu\rangle \left(-\frac{N^2}{144}\right)$$
$$+ \langle\Delta h_{\mu\nu}\rangle\langle h^{\mu\nu} u^2\rangle \left(\frac{N}{72}\right) + \langle\Delta h_{\mu\nu}\rangle\langle\{h^\nu{}_\rho u^\rho u^\mu\}\rangle \left(-\frac{N}{36}\right)$$
$$+ \langle\Delta h_{\mu\nu} u^\mu u_\rho h^{\nu\rho}\rangle \left(-\frac{1}{12}\right) + \langle\Delta h_{\mu\nu} u_\rho u^\mu h^{\nu\rho}\rangle \left(-\frac{1}{12}\right)$$
$$+ \langle\Delta h_{\mu\nu} h^{\mu\nu}\rangle\langle u^2\rangle \left(-\frac{N}{144}\right) + \langle\Delta h_{\mu\nu} h^\mu{}_\rho\rangle\langle u^\nu u^\rho\rangle \left(\frac{N}{36}\right)$$
$$+ \langle\Delta h_{\mu\nu} u^2 h^{\mu\nu}\rangle \left(-\frac{1}{3}\right) + \langle\Delta u^2\rangle\langle h_{\mu\nu} h^{\mu\nu}\rangle \left(\frac{11N}{72}\right)$$
$$+ \langle\{\Delta u_\mu u_\nu h^\mu{}_\rho h^{\nu\rho}\}\rangle \left(\frac{1}{24}\right)$$
$$+ \langle\{\Delta u_\mu u_\nu h^\nu{}_\rho h^{\mu\rho}\}\rangle \left(\frac{1}{24} - \frac{N^2}{144}\right)$$
$$+ \langle\{\Delta u_\mu h^\mu{}_\nu\}\rangle\langle h^\nu{}_\rho u^\rho\rangle \left(-\frac{N}{48}\right)$$
$$+ \langle\{\Delta u_\mu h^\mu{}_\nu h^\nu{}_\rho u^\rho\}\rangle \left(\frac{N^2}{36}\right)$$
$$+ \langle\{\Delta u_\mu h_{\nu\rho}\}\rangle\langle h^{\mu\rho} u^\nu\rangle \left(-\frac{N}{48}\right)$$
$$+ \langle\{\Delta u_\mu h_{\nu\rho}\}\rangle\langle h^{\nu\rho} u^\mu\rangle \left(-\frac{N}{48}\right)$$
$$+ \langle\{\Delta u^2 h_{\mu\nu} h^{\mu\nu}\}\rangle \left(\frac{1}{6} + \frac{23N^2}{288}\right)$$
$$+ \mathrm{i}\langle\Delta u_\mu\rangle\langle[\chi_{+\nu} u^\mu u^\nu]\rangle \left(-\frac{N}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu u_\nu]\rangle\langle\chi_+{}^{\mu} u^\nu\rangle \left(-\frac{N}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu u_\nu u^\mu \chi_+{}^{\nu}]\rangle \left(-\frac{5}{24}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu u_\nu \chi_+{}^{\mu} u^\nu]\rangle \left(\frac{1}{8}\right) + \mathrm{i}\langle[\Delta u_\mu u_\nu \chi_+{}^{\nu} u^\mu]\rangle \left(\frac{1}{8}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu \chi_+{}^{\mu}]\rangle\langle u^2\rangle \left(\frac{1}{4N} - \frac{N}{12}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu \chi_{+\nu}]\rangle\langle u^\mu u^\nu\rangle \left(\frac{1}{2N} + \frac{N}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu u^2]\rangle\langle\chi_+{}^{\mu}\rangle \left(\frac{11N}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u_\mu u^2 \chi_+{}^{\mu}]\rangle \left(\frac{1}{6} + \frac{N^2}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u^2 u_\mu \chi_+{}^{\mu}]\rangle \left(\frac{1}{6} - \frac{7N^2}{48}\right)$$
$$+ \mathrm{i}\langle[\Delta u^2 \chi_{+\mu} u^\mu]\rangle \left(\frac{1}{8} - \frac{N^2}{16}\right)$$
$$+ \mathrm{i}\langle\Delta u_\mu\rangle\langle u^\mu u_\nu\rangle\langle\chi_-{}^{\nu}\rangle \left(-\frac{1}{3}\right)$$
$$+ \mathrm{i}\langle\Delta u_\mu\rangle\langle\chi_-{}^{\mu} u^2\rangle \left(-\frac{2}{3N} + \frac{19N}{72}\right)$$

$$+\,\mathrm{i}\langle\Delta u_\mu\rangle\langle\{\chi_{-\nu}u^\mu u^\nu\}\rangle\left(-\frac{5}{3N}+\frac{11N}{16}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u_\nu\}\rangle\langle\chi_-{}^\mu u^\nu\rangle\left(\frac{1}{3N}+\frac{N}{144}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle\{\chi_+u^\mu u^2\}\rangle\left(-\frac{3}{4N}-\frac{5N}{32}+\frac{N^2}{32}\right)$$
$$+\,\mathrm{i}\langle\Delta u_\mu u_\nu u^\mu\rangle\langle\chi_-{}^\nu\rangle\left(\frac{N}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta u_\mu u_\nu\chi_-{}^\nu u^\mu\rangle\left(-\frac{1}{36}-\frac{N^2}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta u_\mu\chi_-{}^\mu\rangle\langle u^2\rangle\left(-\frac{2}{3N}-\frac{N}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta u_\mu\chi_{-\nu}\rangle\langle u^\mu u^\nu\rangle\left(-\frac{1}{3N}+\frac{N}{18}\right)$$
$$+\,\mathrm{i}\langle\Delta u_\mu\chi_{-\nu}u^\nu u^\mu\rangle\left(-\frac{1}{36}-\frac{N^2}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta\chi_{+\mu}\rangle\langle u^2u^\mu\rangle\left(\frac{N}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta\chi_{+\mu}u^\mu\rangle\langle u^2\rangle\left(-\frac{2}{3N}-\frac{N}{72}\right)$$
$$+\,\mathrm{i}\langle\Delta\chi_{+\mu}u_\nu\rangle\langle u^\mu u^\nu\rangle\left(-\frac{1}{3N}+\frac{N}{18}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u_\nu u^\mu\chi_-{}^\nu\}\rangle\left(\frac{1}{72}-\frac{N^2}{288}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u_\nu\chi_-{}^\mu u^\nu\}\rangle\left(-\frac{1}{12}-\frac{N^2}{288}\right)$$
$$+\,\mathrm{i}\langle\Delta u^2\rangle\langle\chi_{+\mu}u^\mu\rangle\left(\frac{4}{3N}+\frac{11N}{12}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u_\nu\chi_-{}^\nu u^\mu\}\rangle\left(\frac{1}{72}-\frac{5N^2}{288}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu\chi_-{}^\mu\}\rangle\langle u^2\rangle\left(-\frac{N}{144}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu\chi_-{}^\mu u^2\}\rangle\left(-\frac{7}{72}+\frac{N^2}{72}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu\chi_{-\nu}\}\rangle\langle u^\mu u^\nu\rangle\left(-\frac{N}{48}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu\chi_{-\nu}u^\mu u^\nu\}\rangle\left(-\frac{7}{72}+\frac{N^2}{72}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle\chi_+u^\mu u^2\rangle\left(\frac{7}{12N}-\frac{83N}{288}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u^2u_\mu\chi_-{}^\mu\}\rangle\left(-\frac{11}{72}+\frac{23N^2}{96}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u^2\chi_{+\mu}u^\mu\}\rangle\left(-\frac{1}{4}+\frac{77N^2}{288}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u^2\chi_-{}^\mu\}\rangle\left(-\frac{23}{72}-\frac{N^2}{96}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle u^\mu u_\nu\rangle\langle\chi_+u^\nu\rangle\left(-\frac{5}{12}+\frac{N}{8}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle u^2\rangle\langle\chi_+u^\mu\rangle\left(\frac{1}{4}+\frac{N}{8}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle u^2u^\mu\rangle\langle\chi_+\rangle\left(-\frac{3}{8}+\frac{N}{8}\right)$$
$$+\,\mathrm{i}\langle\{\Delta u_\mu u^2\}\rangle\langle\chi_-{}^\mu\rangle\left(-\frac{23N}{144}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle\chi_+u_\nu u^\mu u^\nu\rangle\left(\frac{7}{3N}-\frac{9N}{16}\right)$$
$$+\,\langle\Delta u_\mu\rangle\langle\chi_+u^2u^\mu\rangle\left(\frac{7}{12N}-\frac{83N}{288}\right)$$
$$+\,\langle\Delta u_\mu u_\nu\rangle\langle\chi_+u^\mu u^\nu\rangle\left(-\frac{1}{6N}-\frac{11N}{144}\right)$$
$$+\,\langle\Delta u_\mu u_\nu\rangle\langle\chi_+u^\nu u^\mu\rangle\left(-\frac{1}{6N}-\frac{5N}{144}\right)$$
$$+\,\langle\Delta u_\mu u_\nu u^\mu u^\nu\rangle\langle\chi_+\rangle\left(\frac{N}{24}-\frac{N^2}{16}\right)$$
$$+\,\langle\Delta u_\mu u_\nu\chi_+\rangle\langle u^\mu u^\nu\rangle\left(\frac{1}{12N}-\frac{N}{72}\right)$$
$$+\,\langle\Delta u_\mu u_\nu u^\mu\rangle\langle\chi_+u^\nu\rangle\left(-\frac{23N}{144}\right)$$
$$+\,\langle\Delta u_\mu u_\nu\chi_+u^\mu u^\nu\rangle\left(-\frac{3}{16}+\frac{N^2}{96}\right)$$
$$+\,\langle\Delta u_\mu u_\nu\chi_+u^\nu u^\mu\rangle\left(\frac{83}{144}-\frac{N}{8}+\frac{5N^2}{288}\right)$$
$$+\,\langle\Delta u_\mu\chi_+u_\nu\rangle\langle u^\mu u^\nu\rangle\left(\frac{1}{6N}-\frac{13N}{144}\right)$$
$$+\,\langle\Delta u_\mu\chi_+\rangle\langle u^2u^\mu\rangle\left(-\frac{N}{144}\right)$$
$$+\,\langle\Delta u_\mu\chi_+u^\mu\rangle\langle u^2\rangle\left(\frac{13}{12N}+\frac{N}{144}\right)$$
$$+\,\langle\Delta u_\mu u^2u^\mu\rangle\langle\chi_+\rangle\left(\frac{N}{48}+\frac{N^2}{16}\right)$$
$$+\,\langle\Delta\{u_\mu\chi_+u^2u^\mu\}\rangle\left(\frac{1}{144}+\frac{N^2}{288}\right)$$
$$+\,\langle\Delta u^2\rangle\langle u^2\rangle\langle\chi_+\rangle\left(-\frac{1}{16}+\frac{N}{16}\right)$$
$$+\,\langle\Delta u^2\rangle\langle\chi_+u^2\rangle\left(-\frac{2}{3N}-\frac{13N}{36}+\frac{N^2}{16}\right)$$
$$+\,\langle\Delta u^2u^2\rangle\langle\chi_+\rangle\left(-\frac{5N}{16}+\frac{N^2}{16}\right)$$
$$+\,\langle\Delta u^2\chi_+\rangle\langle u^2\rangle\left(\frac{1}{6N}+\frac{N}{288}\right)$$
$$+\,\langle\Delta u^2\chi_+u^2\rangle\left(-\frac{3}{16}-\frac{N}{16}-\frac{11N^2}{96}\right)$$
$$+\,\langle\Delta\chi_+\rangle\langle u_\mu u_\nu u^\mu u^\nu\rangle\left(-\frac{N}{6}-\frac{N^2}{16}\right)$$

$$
\begin{aligned}
&+ \langle \Delta\chi_+\rangle\langle u^2\rangle\langle u^2\rangle \left(\frac{3}{16}+\frac{N}{32}\right)\\
&+ \langle \Delta\chi_+\rangle\langle u^2u^2\rangle \left(\frac{5N}{12}+\frac{N^2}{16}\right)\\
&+ \langle \Delta\chi_+u_\mu\rangle\langle u^2u^\mu\rangle \left(-\frac{N}{144}\right)\\
&+ \mathrm{i}\langle \Delta u_\mu\rangle\langle h^\mu{}_\nu u^\nu\rangle\langle\chi_-\rangle \left(\frac{1}{8}-\frac{N}{16}\right)\\
&+ \langle \Delta u_\mu\rangle\langle h^\mu{}_{\nu\rho} u^\nu u^\rho\rangle \left(-\frac{N}{8}\right)\\
&+ \mathrm{i}\langle \Delta u_\mu\rangle\langle\{h^\mu{}_\nu u^\nu\chi_-\}\rangle \left(\frac{N}{16}-\frac{N^2}{32}\right)\\
&+ \langle \Delta u_\mu\rangle\langle h^\mu{}_{\nu\rho} u^\nu u^\rho\rangle \left(\frac{N}{4}\right) + \langle \Delta u_\mu\rangle\langle\{h_\nu{}^\mu{}_\rho u^\nu u^\rho\}\rangle \left(\frac{5N}{144}\right)\\
&+ \mathrm{i}\langle \Delta u_\mu u_\nu\rangle\langle h^{\mu\nu}\chi_-\rangle \left(-\frac{3N}{16}\right)\\
&+ \langle \Delta u_\mu u_\nu\rangle\langle h^{\mu\nu}{}_\rho u^\rho\rangle \left(-\frac{N}{144}\right)\\
&+ \langle \Delta u_\mu u_\nu\rangle\langle h^{\nu\mu}{}_\rho u^\rho\rangle \left(-\frac{N}{144}\right)\\
&+ \langle \Delta u_\mu u_\nu\rangle\langle h_\rho{}^{\mu\nu} u^\rho\rangle \left(-\frac{N}{24}\right)\\
&+ \mathrm{i}\langle \Delta u_\mu h^\mu{}_\nu u^\nu\rangle\langle\chi_-\rangle \left(\frac{N}{8}\right)\\
&+ \langle \Delta h_{\mu\nu\rho}\rangle\langle\{u^\mu u^\rho u^\nu\}\rangle \left(-\frac{N}{18}\right)\\
&+ \langle\{\Delta u_\mu u_\nu u_\rho h^{\mu\nu\rho}\}\rangle \left(\frac{1}{24}\right)\\
&+ \langle\{\Delta u_\mu u_\nu u_\rho h^{\nu\mu\rho}\}\rangle \left(-\frac{1}{24}-\frac{N^2}{144}\right)\\
&+ \langle\{\Delta u_\mu u_\nu u_\rho h^{\rho\mu\nu}\}\rangle \left(-\frac{1}{12}\right)\\
&+ \mathrm{i}\langle\{\Delta u_\mu u_\nu h^{\mu\nu}\}\rangle\langle\chi_-\rangle \left(\frac{N}{32}\right)\\
&+ \langle\{\Delta u_\mu u_\nu h^{\mu\nu}{}_\rho u^\rho\}\rangle \left(\frac{1}{24}+\frac{N^2}{48}\right)\\
&+ \langle\{\Delta u_\mu u_\nu h^{\nu\mu}{}_\rho u^\rho\}\rangle \left(\frac{1}{24}\right)\\
&+ \mathrm{i}\langle\{\Delta u_\mu u_\nu \chi_- h^{\mu\nu}\}\rangle \left(-\frac{1}{8}+\frac{3N}{32}\right)\\
&+ \langle\{\Delta u_\mu h_\nu{}^\mu{}_\rho\}\rangle\langle u^\nu u^\rho\rangle \left(-\frac{N}{48}\right)\\
&+ \langle\{\Delta u_\mu h_\nu{}^\mu{}_\rho\}\rangle\langle u^\nu u^\rho\rangle \left(\frac{N}{144}\right)\\
&+ \mathrm{i}\langle \Delta\chi_-\rangle\langle h_{\mu\nu}u^\mu u^\nu\rangle \left(\frac{N}{16}\right)\\
&+ \langle \Delta h_{\mu\nu\rho}\rangle\langle u^\mu u^\nu u^\rho\rangle \left(-\frac{N}{18}\right)\\
&+ \mathrm{i}\langle\{\Delta u_\mu\chi_- u_\nu h^{\mu\nu}\}\rangle \left(\frac{1}{48}-\frac{N}{32}\right)\\
&+ \mathrm{i}\langle\{\Delta u_\mu\chi_- h^\mu{}_\nu u^\nu\}\rangle \left(-\frac{1}{48}+\frac{N}{32}\right)\\
&+ \mathrm{i}\langle\{\Delta \chi_- u_\mu u_\nu h^{\mu\nu}\}\rangle \left(\frac{1}{6}-\frac{N^2}{96}\right)\\
&+ \mathrm{i}\langle\{\Delta \chi_- u_\mu h^\mu{}_\nu u^\nu\}\rangle \left(-\frac{3}{16}-\frac{3N}{32}+\frac{N^2}{48}\right)\\
&+ \mathrm{i}\langle\{\Delta \chi_- h_{\mu\nu}\}\rangle\langle u^\mu u^\nu\rangle \left(-\frac{N}{32}\right)\\
&+ \mathrm{i}\langle\{\Delta \chi_- h_{\mu\nu}u^\mu u^\nu\}\rangle \left(\frac{7}{48}+\frac{N^2}{48}\right). \qquad \text{(C.1)}
\end{aligned}
$$

$\langle\cdot\rangle$ denotes the trace in $SU(N)$. We calculated the generic case with N flavors in order to make the above counterterm also usable for the quenched case [24,25]. The calculation was performed in euclidean space-time, but above we provide the result transformed back to Minkowski space. Furthermore we used the notation

$$
\begin{aligned}
[A_1...A_n] &:= A_1A_2...A_n - A_nA_{n-1}...A_1;\\
\{A_1...A_n\} &:= A_1A_2...A_n + A_nA_{n-1}...A_1.
\end{aligned}
$$

The above expression corresponds only to the part of $\mathcal{A}_2^{(2)}$, which does not involve any external vector sources and can contribute to processes involving four pseudo Goldstone fields, i.e. can be used for the (physical) $K\to\pi\pi$ and $K\to\pi\pi\pi$ amplitudes. For obvious reasons we do not provide a minimal basis for $\mathcal{L}_{\mathrm{w}}^{(2)}$. However, for the part of $\mathcal{A}_2^{(2)}$ shown here, all operators are linearly independent: One can use Cayley–Hamilton relations (CHR), the Bianchi identities and partial integrations to (potentially) reduce the number of operators and find a basis. Yet, the CHR are not usable since we worked in general $SU(N)$, the Bianchi identities involve vector sources which were neglected, and partial integrations cannot be employed since it would necessarily generate an operator involving Δ_μ [2].

We do not present the full result, including also the operators containing Δ_μ and $f_\pm^{\mu\nu}$, here, since it is approximately a factor four times larger. It can however be obtained from the author.

Appendix D: Notation and definitions

The notation is as follows:

$$
U := u^2 := \exp\left(\frac{\mathrm{i}\sqrt{2}\phi}{F}\right) \quad ; \quad U \longrightarrow g_{\mathrm{R}} U g_{\mathrm{L}}^\dagger,
$$

with ϕ, defined in (2.4), representing the pseudo Goldstone bosons and the arrow showing the response of U to a chiral transformation $(g_{\mathrm{L}}, g_{\mathrm{R}}) \in \big(SU(3)_{\mathrm{L}}, SU(3)_{\mathrm{R}}\big)$.

[2] In a full basis it would however be preferable to trade operators within $\tilde{\mathcal{A}}_2^{(2)}$ for operators involving Δ_μ.

The building blocks used are:

$$
\begin{aligned}
\Delta &:= u\lambda_6 u^\dagger,\\
u_\mu &:= \mathrm{i}\big(u^\dagger(\partial_\mu - \mathrm{i}r_\mu)u - u(\partial_\mu - \mathrm{i}l_\mu)u^\dagger\big),\\
\chi_\pm &:= u^\dagger \chi u^\dagger \pm u\chi^\dagger u,\\
\Gamma_\mu &:= \frac{1}{2}\big(u^\dagger(\partial_\mu - \mathrm{i}r_\mu)u + u(\partial_\mu - \mathrm{i}l_\mu)u^\dagger\big),\\
\nabla_\mu X &:= \partial_\mu X + [\Gamma_\mu, X],\\
\Delta_\mu &:= \nabla_\mu \Delta + \frac{\mathrm{i}}{2}\{\Delta, u_\mu\},\\
u_{\mu\nu} &:= \nabla_\mu u_\nu,\\
h_{\mu\nu} &:= \nabla_\mu u_\nu + \nabla_\nu u_\mu,\\
h_{\mu\nu\rho} &:= \nabla_\mu h_{\nu\rho},\\
\chi_{\pm,\mu} &:= \nabla_\mu \chi_\pm - \frac{\mathrm{i}}{2}\{\chi_\mp, u_\mu\},\\
\chi_{\pm\mu} &:= \nabla_\mu \chi_\pm,\\
\chi_{\pm\{\mu,\nu\}} &:= \{\nabla_\mu, \nabla_\nu\}\chi_\pm .
\end{aligned}
\tag{D.1}
$$

All of these transform like:

$$X \to h(\phi, g) X h^\dagger(\phi, g),$$

where $h(\phi, g)$ is the compensating $SU(N)_V$ transformation defined by:

$$u \to g_{\mathrm{R}} u h^\dagger(\phi, g) = h(\phi, g) u g_{\mathrm{L}}^\dagger,$$

with the exception of Δ_μ which transforms like $\Delta_\mu \to g_{\mathrm{L}} \Delta_\mu g_{\mathrm{L}}^\dagger$. In addition we used notation that derives from the above ($u^2 := u_\mu u^\mu$, $h_{\mu\nu\rho\sigma} := \nabla_\mu \nabla_\nu h_{\rho\sigma}$ etc.). All calculations were performed with FORM 3.1 [20].

Furthermore, we used:

$$\varepsilon := 4 - d \;\; ; \quad \hat{N} := (4\pi)^{-2} \;\; ; \quad \Lambda := \frac{\hat{N}}{\varepsilon}. \tag{D.2}$$

Appendix E: Expansion of the building blocks in the ξ fields

Below we provide the expansion of the building blocks given in Appendix D in terms of the quantum fluctuation fields ξ. The building blocks on the RHS are evaluated at the EOM (expressed by the bar).

$$
\begin{aligned}
\Delta &= \bar{\Delta} - \frac{\mathrm{i}}{2}[\bar{\Delta}, \xi] - \frac{1}{8}[[\bar{\Delta}, \xi], \xi]\\
&\quad + \frac{\mathrm{i}}{48}[[[\bar{\Delta}, \xi], \xi], \xi] + \mathcal{O}(\xi^4),\\
u_\mu &= \bar{u}_\mu - \xi_\mu - \frac{1}{8}[[\bar{u}_\mu, \xi], \xi] + \frac{1}{24}[[\xi_\mu, \xi], \xi]\\
&\quad + \mathcal{O}(\xi^4),\\
\Gamma_\mu &= \bar{\Gamma}_\mu + \frac{1}{4}[\bar{u}_\mu, \xi] - \frac{1}{8}[\xi_\mu, \xi]\\
&\quad - \frac{1}{96}[[[\bar{u}_\mu, \xi], \xi], \xi] + \mathcal{O}(\xi^4),\\
\chi_\pm &= \bar{\chi}_\pm - \frac{\mathrm{i}}{2}\{\bar{\chi}_\mp, \xi\} - \frac{1}{8}\{\{\bar{\chi}_\pm, \xi\}, \xi\}\\
&\quad + \frac{\mathrm{i}}{48}\{\{\{\bar{\chi}_\mp, \xi\}, \xi\}, \xi\} + \mathcal{O}(\xi^4),\\
f_\pm^{\mu\nu} &= \bar{f}_\pm^{\mu\nu} - \frac{\mathrm{i}}{2}[\bar{f}_\mp^{\mu\nu}, \xi] - \frac{1}{8}[[\bar{f}_\pm^{\mu\nu}, \xi], \xi]\\
&\quad + \frac{\mathrm{i}}{48}[[[\bar{f}_\mp^{\mu\nu}, \xi], \xi], \xi] + \mathcal{O}(\xi^4),\\
\Delta_\mu &= \bar{\Delta}_\mu - \frac{\mathrm{i}}{2}[\bar{\Delta}_\mu, \xi] - \frac{1}{8}[[\bar{\Delta}_\mu, \xi], \xi]\\
&\quad + \frac{\mathrm{i}}{48}[[[\bar{\Delta}_\mu, \xi], \xi], \xi] + \mathcal{O}(\xi^4),\\
\chi_{\pm,\mu} &= \bar{\chi}_{\pm,\mu} - \frac{\mathrm{i}}{2}\{\bar{\chi}_{\mp,\mu}, \xi\} - \frac{1}{8}[[\bar{\chi}_{\pm,\mu}, \xi], \xi] + \mathcal{O}(\xi^3),\\
u_{\mu\nu} &= \bar{u}_{\mu\nu} - \xi_{\mu\nu} + \frac{1}{4}[[\bar{u}_\mu, \xi], \bar{u}_\nu] - \frac{1}{8}[[\bar{u}_{\mu\nu}, \xi], \xi]\\
&\quad - \frac{1}{4}[[\bar{u}_\mu, \xi], \xi_\nu] - \frac{1}{4}[[\bar{u}_\nu, \xi], \xi_\mu] + \mathcal{O}(\xi^3),\\
\hat{X} &= -\xi_{\mu\mu} + \frac{1}{4}[[\bar{u}_\mu, \xi], \bar{u}_\mu] + \frac{1}{4}\{\bar{\chi}_+, \xi\}\\
&\quad - \frac{1}{2N}\langle \bar{\chi}_+ \xi \rangle - \frac{1}{2}[[\bar{u}_\mu, \xi], \xi_\mu] - \frac{1}{4}\xi \bar{\chi}_- \xi\\
&\quad + \frac{1}{4N}\langle \bar{\chi}_- \xi^2 \rangle + \mathcal{O}(\xi^3).
\end{aligned}
\tag{E.1}
$$

We use the conventions $u = \bar{u}\exp(\mathrm{i}\xi/2)$, $\xi = \sum \lambda_a \xi_a$, $\langle \lambda_a \lambda_b \rangle = 2\delta_{ab}$ and the notation $\xi_\mu := \nabla_\mu \xi$, $\xi_{\mu\nu} := \nabla_\mu \nabla_\nu \xi$.

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